143. On the Inductive Dimension of Product Spaces

By Nobuo KIMURA

Hokkaido Gakugei University

(Comm. by Kinjirô KUNUGI, M.J.A., Nov. 12, 1963)

As is well known, the *large inductive dimension* of a topological space X, denoted by $\operatorname{Ind} X$, is defined as follows. In case $X=\phi$, we put $\operatorname{Ind} X=-1$, and define $\operatorname{Ind} X \leqq n$ for $n \geqq 0$ inductively by the requirement that for any pair of a closed set F and an open set G with $F \subset G$ there exists an open set U such that $F \subset U \subset G$, $\operatorname{Ind}(\overline{U}-U) \leqq n-1$. $\operatorname{Ind} X=n$ means that we have $\operatorname{Ind} X \leqq n$ but not $\operatorname{Ind} X \leqq n-1$.

E. Čech [1] proved that the subset theorem and the sum theorem hold for the large inductive dimension of perfectly normal spaces. C. H. Dowker [2] generalized Čech's results mentioned above by proving that the subset theorem and the sum theorem hold still for the large inductive dimension of totally normal spaces. Here a normal space X is said to be *totally normal* (Dowker [2]) if each open subspace of X has a locally finite open covering by open subsets each of which is an F_σ set of X. Since every perfectly normal space is totally normal ([2]) Čech's results are included in Dowker's results.

As for the large inductive dimension of product spaces, in 1960 K. Nagami [6] proved the validity of the inequality

$$\operatorname{Ind}(X \times Y) \leqq \operatorname{Ind} X + \operatorname{Ind} Y$$

for the case where X is a perfectly normal, paracompact space and Y is a metrizable space. This seems to be the most general result known hitherto.

In the present note we shall establish that the above inequality holds still for the case where $X \times Y$ is a countably paracompact, totally normal space and Y is a metrizable space; this is stated as Theorem 4 below. If X is a perfectly normal space and Y a metrizable space, then $X \times Y$ is also perfectly normal as was proved in Morita [4] and hence $X \times Y$ is totally normal and countably paracompact. Thus Nagami's result is contained in our Theorem 4.

Our proof of Theorem 4 is based on two theorems; one is a theorem of K. Morita [5] on product spaces and the other is a generalized sum theorem which will be proved below as Theorem 3.

Our Theorem 3, which seems to be of some interest in itself, asserts that if $\{A_\alpha\}$ is a locally finite closed covering of a countably paracompact, totally normal space X and if $\operatorname{Ind} A_\alpha \leqq n$ for each α then $\operatorname{Ind} X \leqq n$.

1. We can easily prove the following

Lemma 1. *Let Y be a metric space with* Ind $Y \leqq n$. *Then there is a countable family* $\{\mathfrak{V}_i \mid i=1, 2, \cdots\}$ *of locally finite open coverings* $\mathfrak{V}_i = \{V_{i\alpha} \mid \alpha \in \Omega_i\}$ *of Y such that* Ind $\mathfrak{B}_Y(V_{i\alpha}) \leqq n-1$, *and the diameter of $V_{i\alpha}$ is smaller than 2^{-i} for $\alpha \in \Omega_i$. Here $\mathfrak{B}_Y(V_{i\alpha})$ means the boundary of $V_{i\alpha}$.*

Let us put

$$W(\alpha_1, \cdots, \alpha_i) = V_{1\alpha_1} \cap \cdots \cap V_{i\alpha_i};$$

then the following theorem can be proved without the dimensional condition.

Theorem 1. ([5, Theorem 2.3]). *$X \times Y$ is countably paracompact and normal if and only if* (i) *X is countably paracompact and normal, and* (ii) *for any family* $\{G(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, i=1, 2, \cdots\}$ *of open sets of X such that* $\{G(\alpha_1, \cdots, \alpha_i) \times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, i=1, 2, \cdots\}$ *is an open covering of $X \times Y$ and* $G(\alpha_1, \cdots, \alpha_i) \subset G(\alpha_1, \cdots, \alpha_i, \alpha_{i+1})$, *there is a family* $\{F(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu\}$ *of closed sets of X such that* $\{F(\alpha_1, \cdots, \alpha_i) \times W(\alpha_1, \cdots, \alpha_i) \mid \alpha_\nu \in \Omega_\nu, i=1, 2, \cdots\}$ *is a covering of $X \times Y$.*

2. The following Theorems 2 and 3 should be compared with [2, Prop. 2.1] and [2, Theorem 4].

Theorem 2. *Let X be totally normal and countably paracompact, and let Ω be a well ordered set. We suppose that $\{X_\alpha \mid \alpha \in \Omega\}$ is a family of open sets of X having the following properties*:
(i) $X_\alpha \subset X_\beta$ *if* $\alpha > \beta$, (ii) $\bigcup_{\alpha \in \Omega} X_\alpha = X$, (iii) $\bigcap_{\alpha \in \Omega} X_\alpha = \phi$, (iv) Ind$(X_\alpha - X_{\alpha+1}) \leqq n$, (v) $\{X_\alpha - X_{\alpha+1} \mid \alpha \in \Omega\}$ *is locally finite. Then* Ind $X \leqq n$.

Before the proof we shall give some notations and two lemmas.

Put $D_\alpha = X_\alpha - X_{\alpha+1}$. $\{D_\alpha \mid \alpha \in \Omega\}$ is a disjoint family. Then we can define subsets D^i $(i=1, 2, \cdots)$ of X as follows. A point x of X belongs to D^i if and only if $\{\alpha \mid V(x) \cap D_\alpha \neq \phi, \alpha \in \Omega\}$ consists of at least i elements for any neighborhood $V(x)$ and consists of exactly i elements for some neighborhood $V(x)$. Then we have a disjoint union $X = D^1 \cup D^2 \cup \cdots$. Let $D^i_\alpha = D^i \cap D_\alpha$. According to (iv) and the subset theorem we have Ind $D^i_\alpha \leqq n$.

Lemma 2. Ind $D^i \leqq n$.

Proof. $D^i = \bigcup_{\alpha \in \Omega} D^i_\alpha$ is a disjoint union. For any point x of D^i_γ $(\gamma \in \Omega)$ there is some neighborhood $U_\gamma(x)$ of x such that $\{\alpha \mid U_\gamma(x) \cap D_\alpha \neq \phi, \alpha \in \Omega\}$ consists of exactly i elements. Let those elements be $\alpha_1, \alpha_2, \cdots, \alpha_i$. If $\gamma > \alpha_1$, since $D_{\alpha_1} = X_{\alpha_1} - X_{\alpha_1+1}$, we have $D_{\alpha_1} \cap X_\gamma = \phi$ by (i). Hence $V_\gamma(x) \cap D_{\alpha_1} = \phi\}$ where $U_\gamma(x) \cap x_\gamma = V_\gamma(x)$. This means that $\{\alpha \mid V_\gamma(x) \cap D_\alpha \neq \phi\}$ has at most $i-1$ elements $\alpha_2, \alpha_3, \cdots, \alpha_i$. This contradicts the assumption that $x \in D^i_\gamma$. Thus $\alpha_1 \geqq \gamma$. In the same way we have $\alpha_2 \geqq \gamma, \cdots, \alpha_i \geqq \gamma$. Hence we can assume without loss of generality that $\gamma = \alpha_1 < \alpha_2 < \cdots < \alpha_i$.

Take an arbitrary point y of $U_\gamma(x)\cap D^i$. If $y\notin D^i_\gamma$, $y\in D^i_{\alpha_j}\subset D_{\alpha_j}$ for some $j(1<j\leqq i)$. Then there is some neighborhood $V(y)$ of y such that $V(y)\cap D_\gamma=\phi$. Thus $U_\gamma(x)\cap V(y)$ intersects at most $i-1$ D_α's. This shows that $y\notin D^i$, which contradicts $y\in U_\gamma(x)\cap D^i$. Therefore $y\in D^i_\gamma$. Hence $U_\gamma(x)\cap D^i\subset D^i_\gamma$. Thus D^i_γ is open in D^i. Since $\{D^i_\alpha \mid \alpha\in\Omega\}$ is a mutually disjoint family, D^i_γ is also closed in D^i. Now Ind $D^i\leqq n$ follows from [2, Prop. 5.1].

Lemma 3. $\bigcup_{i=1}^{r} D^i$ *is open in* X.

Proof. Suppose that $x\in\bigcup_{i=1}^{r} D^i$. Then $x\in D^j$ for some $j(1\leqq j\leqq r)$, and hence there is a neighborhood $U(x)$ of x such that $\{\alpha \mid U(x)\cap D_\alpha \neq\phi\}$ has i elements. Since $U(x)$ is also a neighborhood of its element y, we have $y\in D^k$ for some $k(k\leqq j)$. Then $y\in\bigcup_{i=1}^{r} D^i$; i.e., $U(x)\subset\bigcup_{i=1}^{r} D^i$. Thus Lemma 3 is proved.

Proof of Theorem 2. Let $D^1\cup D^2\cup\cdots\cup D^k=Z_k$. Suppose that Ind $Z_{k-1}\leqq n$. From [2, Prop. 4.7] Z_k is totally normal. Since Z_{k-1} is open in Z_k by Lemma 3, D^k is closed in Z_k. Hence, by [2, Theorem 3], we have Ind $Z_k\leqq n$. Since we have clearly Ind $Z_1=$ Ind $D^1\leqq n$, by induction on k we can conclude that Ind $Z_k\leqq n$ for any k. Now $Z_1\subset Z_2\subset\cdots\subset X$ and $X=\bigcup_{i=1}^{\infty} Z_i$. Since X is countably paracompact and normal, there is a family of closed sets $\{F_i\}$ such that $F_i\subset Z_i$ and $X=\bigcup_{i=1}^{\infty} F_i$. By the subset theorem Ind $F_i\leqq n$. Hence we have Ind $X\leqq n$ by the sum theorem.

Theorem 3. (The generalized sum theorem.) *Suppose that X is totally normal and countably paracompact. Let Ω be a well ordered set. If $\{A_\alpha \mid \alpha\in\Omega\}$ is a locally finite closed covering of X and if* Ind $A_\alpha\leqq n$ *for each $\alpha\in\Omega$, then* Ind $X\leqq n$.

Proof. We put $X_\alpha=X-\bigcup_{\beta<\alpha} A_\beta$ and $D_\alpha=A_\alpha-\bigcup_{\beta<\alpha} A_\beta$; then $X_\alpha\supset X_{\alpha+1}$ and $\bigcap_{\alpha\in\Omega} X_\alpha=X-\bigcup_{\alpha\in\Omega} A_\alpha=\phi$. Since $\{A_\alpha \mid \alpha\in\Omega\}$ is locally finite, $\bigcup_{\beta<\alpha} A_\beta$ is closed in X. Hence D_α is open in A_α and X_α is open in X. Clearly we have Ind $D_\alpha\leqq$ Ind $A_\alpha\leqq n$. $\{D_\alpha\}$ is mutually disjoint. By definition $X_\alpha=\bigcup_{\gamma\geqq\alpha} D_\gamma=D_\alpha\cup X_{\alpha+1}$. Thus $D_\alpha=X_\alpha-X_{\alpha+1}$ and Ind $(X_\alpha-X_{\alpha+1})\leqq n$. Now Theorem 2 is applicable to the present case, and we have Ind $X\leqq n$. This proves Theorem 3.

3. Now we are in a position to prove our main theorem.

Theorem 4. *If Y is a metric space and if $X\times Y$ is totally normal and countably paracompact, then*

$$\text{(1)} \qquad \operatorname{Ind}(X\times Y)\leqq \operatorname{Ind} X+\operatorname{Ind} Y.$$

(*Here we assume that at least one of X, Y is not empty.*)

Proof. If Ind $Y=-1$ then (1) is always true. We shall prove the inequality (1) by induction on $n=\operatorname{Ind} Y$. For this purpose, let us assume that (1) is true if $\operatorname{Ind} Y\leqq n-1$. Now assume that $\operatorname{Ind} Y\leqq n$. We want to show that (1) is true in this case.

Select a countable family $\mathfrak{V}_i=\{V_{i\alpha}|\alpha\in\Omega_i\}$ $(i=1,2,\cdots)$ of open coverings of Y satisfying the conditions of Lemma 1.

If $\operatorname{Ind} X=-1$ then (1) is trivially true. Assume that (1) is true in case $\operatorname{Ind} X\leqq m-1$, and $\operatorname{Ind} Y\leqq n$; we refer to this as the second induction hypothesis. Suppose that $\operatorname{Ind} X\leqq m$.

Let F be a closed subset of $X\times Y$ and G be an open subset of $X\times Y$ such that $F\subset G$. There exist two open sets L, M of $X\times Y$ such that $F\subset M\subset\overline{M}\subset L\subset\overline{L}\subset G$. We put $N_1=X\times Y-\overline{M}$, $N_2=L$. Then $\mathfrak{N}=\{N_1, N_2\}$ is an open covering of $X\times Y$.

We put $G(\alpha_1,\cdots,\alpha_i;k)=\operatorname{Int}\{x|x\times W(\alpha_1,\cdots,\alpha_i)\subset N_k\}$ $(k=1,2)$. (Here $\operatorname{Int} A$ means the interior of the subset A.) Then $G(\alpha_1,\cdots,\alpha_i;k)\times W(\alpha_1,\cdots,\alpha_i)\subset N_k$.

Let us put $G'(\alpha_1,\cdots,\alpha_i;k)=\bigcup_{j\leqq i} G(\alpha_1,\cdots,\alpha_j;k)$; then $G'(\alpha_1,\cdots,\alpha_i;k)\subset G'(\alpha_1,\cdots,\alpha_i,\alpha_{i+1};k)$ and $G'(\alpha_1,\cdots,\alpha_i;k)\times W(\alpha_1,\cdots,\alpha_i)\subset N_k$.

Set $G(\alpha_1,\cdots,\alpha_i)=G'(\alpha_1,\cdots,\alpha_i;1)\cup G'(\alpha_1,\cdots,\alpha_i;2)$; then $G(\alpha_1,\cdots,\alpha_i)\subset G(\alpha_1,\cdots,\alpha_i,\alpha_{i+1})$. Now

$$\{G(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)|\alpha_\nu\in\Omega_\nu,\ i=1,2,\cdots\} \tag{2}$$

is an open covering of $X\times Y$. For, if $(x,y)\in X\times Y$, there is some k(1 or 2) such that $(x,y)\in N_k$. Then there are a neighborhood $U(x)$ of x and elements $\alpha_1,\alpha_2,\cdots,\alpha_i$ of Ω such that $U(x)\times W(\alpha_1,\cdots,\alpha_i)\subset N_k$ $(y\in W(\alpha_1,\cdots,\alpha_i))$. Thus $U(x)\subset G(\alpha_1,\cdots,\alpha_i;k)$, and $(x,y)\in G(\alpha_1,\cdots\alpha_i;k)\times W(\alpha_1,\cdots,\alpha_i)\subset G(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)$. Therefore (2) is an open covering.

Now Theorem 1 is applicable to (2). Hence there exists a family $\{F(\alpha_1,\cdots,\alpha_i)|\alpha_\nu\in\Omega_\nu,\ i=1,2,\cdots\}$ of closed subsets of X such that $F(\alpha_1,\cdots,\alpha_i)\subset G(\alpha_1,\cdots,\alpha_i)$ and such that $\{F(\alpha_1,\cdots,\alpha_i)\times W(\alpha_1,\cdots,\alpha_i)|\alpha_\nu\in\Omega_\nu,\ i=1,2,\cdots\}$ is a covering of $X\times Y$. From the relation $F(\alpha_1,\cdots,\alpha_i)\subset\bigcup_{k=1}^{2} G'(\alpha_1,\cdots,\alpha_i;k)$ it follows that there exist closed sets $F(\alpha_1,\cdots,\alpha_i;k)$ of X such that $F(\alpha_1,\cdots,\alpha_i)=\bigcup_{k=1}^{2} F(\alpha_1,\cdots,\alpha_i;k)$, $F(\alpha_1,\cdots,\alpha_i;k)\subset G'(\alpha_1,\cdots,\alpha_1;k)$.

By the assumption that $\operatorname{Ind} X\leqq m$ there exist open subsets $H(\alpha_1,\cdots,\alpha_i;k)$ of X such that $\operatorname{Ind}\mathfrak{B}_X(H(\alpha_1,\cdots,\alpha_i;k))\leqq m-1$ and $F(\alpha_1,\cdots,\alpha_i;k)\subset H(\alpha_1,\cdots,\alpha_i;k)\subset G'(\alpha_1,\cdots,\alpha_i;k)$.

Since $\operatorname{Ind}\mathfrak{B}_Y(V_{i\alpha})\leqq n-1$ and $\mathfrak{B}_Y(\bigcap_{j=1}^{i} V_{j\alpha_j})\subset\bigcup_{j=1}^{i}\mathfrak{B}_Y(V_{j\alpha_j})$, we have $\operatorname{Ind}\mathfrak{B}_Y(W(\alpha_1,\cdots,\alpha_i))\leqq n-1$ as a consequence of the sum theorem.

Now $\mathfrak{B}_{X\times Y}(H(\alpha_1,\cdots,\alpha_i;k)\times W(\alpha_1,\cdots,\alpha_i))=[\mathfrak{B}_X(H(\alpha_1,\cdots,\alpha_i;k))\times\overline{W(\alpha_1,\cdots,\alpha_i)}]\cup[\overline{H(\alpha_1,\cdots,\alpha_i;k)}\times\mathfrak{B}_Y(W(\alpha_1,\cdots,\alpha_i))]$. According to the second induction hypothesis $\operatorname{Ind}(\mathfrak{B}_X(H(\alpha_1,\cdots,\alpha_i;k)\times\overline{W(\alpha_1,\cdots,\alpha_i)})$

$\leqq \mathrm{Ind}\, \mathfrak{B}_X(H(\alpha_1,\cdots,\alpha_i;k))+\mathrm{Ind}\, \overline{W(\alpha_1,\cdots,\alpha_i)} \leqq (m-1)+n=m+n-1.$
Similarly by the first induction hypothesis

$$\mathrm{Ind}\,(\overline{H(\alpha_1,\cdots,\alpha_i;k)}\times \mathfrak{B}_Y(W(\alpha_1,\cdots,\alpha_i)) \leqq m+(n-1)=m+n-1.$$

Applying the sum theorem, we have

(3) $\mathrm{Ind}\, \mathfrak{B}_{X\times Y}(H(\alpha_1,\cdots,\alpha_i;k)\times W(\alpha_1,\cdots,\alpha_i)) \leqq m+n-1 \quad (k=1,2).$

On the other hand, the family $\{H(\alpha_1,\cdots,\alpha_i;k)\times W(\alpha_1,\cdots,\alpha_i) \mid \alpha_\nu \in \Omega_\nu;\ i=1,2,\cdots;\ k=1,2\}$ is an open covering of $X\times Y$ and is a refinement of $\mathfrak{N}$.

Let us put $H_i=\bigcup\{H(\alpha_1,\cdots,\alpha_i;2)\times W(\alpha_1,\cdots,\alpha_i) \mid \alpha_\nu\in\Omega_\nu\}$, $K_i=\bigcup\{H(\alpha_1,\cdots,\alpha_i;1)\times W(\alpha_1,\cdots,\alpha_i)\mid \alpha_\nu\in\Omega_\nu\}$, and $P_1=H_1$, $Q_1=K_1-\overline{H}_1$, $P_i=H_i-\bigcup_{j=1}^{i-1}\overline{K}_j$, $Q_i=K_i-\bigcup_{j=1}^{i}\overline{H}_j$ $(i\geqq 2)$, $P=\bigcup_{i=1}^{\infty}P_i$, $Q=\bigcup_{i=1}^{\infty}Q_i$.[1)]

Then we have

(4) $$X\times Y=(\bigcup_{i=1}^{\infty}\overline{P}_i)\cup(\bigcup_{i=1}^{\infty}\bar{Q}_i),$$

(5) $$P\cap Q=\phi,\ \overline{P}_j\subset G\ (j=1,2,\cdots),\ Q\cap\overline{M}=\phi.$$

Finally we put $V=X\times Y-\bar{Q}$. Since $Q\cap M=\phi$ by (5) and M is open, we have $\bar{Q}\cap M=\phi$ and hence $F\subset M\subset V$.

On the other hand, since $V=X\times Y-\bar{Q}\subset X\times Y-\bigcup_{i=1}^{\infty}\bar{Q}_i\subset\bigcup_{i=1}^{\infty}\overline{P}_i\subset G$ by (4) and (5), we have

(6) $$F\subset V\subset G.$$

Since $\overline{P}_i=P_i\cup(\overline{P}_i-P_i)$, $\bar{Q}_i=Q_i\cup(\bar{Q}_i-Q_i)$, we have from (4)

(7) $$X\times Y=P\cup Q\cup(\bigcup_{i=1}^{\infty}(\overline{P}_i-P_i))\cup(\bigcup_{i=1}^{\infty}(\bar{Q}_i-Q_i)).$$

From (5) and the openness of P it follows that $P\cap\bar{Q}=\phi$. Hence $P\cap(\bar{Q}-Q)=\phi$. Therefore we have by (7)

(8) $$\bar{Q}-Q\subset\bigcup_{i=1}^{\infty}(\overline{P}_i-P_i)\cup(\bigcup_{i=1}^{\infty}(\bar{Q}_i-Q_i)).$$

Since $\{W(\alpha_1,\cdots,\alpha_i)\mid\alpha_\nu\in\Omega_\nu\}$ is locally finite, we have

$$\overline{H}_i-H_i\subset\bigcup_{\alpha}\{\mathfrak{B}_{X\times Y}(H(\alpha_1,\cdots,\alpha_i;2)\times W(\alpha_1,\cdots,\alpha_i))\mid\alpha_\nu\in\Omega_\nu\},$$

$$\overline{K}_i-K_i\subset\bigcup_{\alpha}\{\mathfrak{B}_{X\times Y}(H(\alpha_1,\cdots,\alpha_i;1)\times W(\alpha_1,\cdots,\alpha_i))\mid\alpha_\nu\in\Omega_\nu\}.$$

Since $\{\mathfrak{B}_{X\times Y}(H(\alpha_1,\cdots,\alpha_i;k)\times W(\alpha_1,\cdots,\alpha_i))\mid\alpha_\nu\in\Omega_\nu\}$ is a locally finite family of closed sets, from (3) and Theorem 3, if follows that

$$\mathrm{Ind}\,(\bigcup_{\alpha}\{\mathfrak{B}_{X\times Y}(H(\alpha_1,\cdots,\alpha_i;k)\times W(\alpha_1,\cdots,\alpha_i))\mid\alpha_\nu\in\Omega_\nu\})\leqq m+n-1.$$

Hence $\mathrm{Ind}\,(\overline{H}_i-H_i)\leqq m+n-1$, $\mathrm{Ind}\,(\overline{K}_i-K_i)\leqq m+n-1$. Since $X\times Y$ is totally normal, we have, by the sum theorem,

$$\mathrm{Ind}\,(\overline{P}_i-P_i)\leqq m+n-1,\ \mathrm{Ind}\,(\bar{Q}_i-Q_i)\leqq m+n-1.$$

By applying the sum theorem again we have from (8) $\mathrm{Ind}\,(\bar{Q}-Q)\leqq m+n-1$, Hence

(9) $$\mathrm{Ind}\,(\overline{V}-V)\leqq m+n-1.$$

1) The argument below is the same as that in [3, Lemma 2.2].

The relation (9) together with (6) shows that $\mathrm{Ind}\,(X\times Y)\leqq m+n$. Thus (1) holds for X with $\mathrm{Ind}\,X\leqq m$. Therefore the inequality (1) for any X and for any Y with $\mathrm{Ind}\,Y\leqq n$ is proved under the first induction hypothesis. Consequently, the proof of Theorem 4 is completed.

4. K. Morita [4] proved that if X is perfectly normal and Y is metrizable then $X\times Y$ is perfectly normal. Since any perfectly normal space is totally normal and countably paracompact, the following theorem follows directly from Theorem 4.

Theorem 5. *If X is a perfectly normal space and Y is a metrizable space then* $\mathrm{Ind}\,(X\times Y)\leqq \mathrm{Ind}\,X+\mathrm{Ind}\,Y$.

References

[1] E. Čech: Sur la dimension des espaces parfaitement normaux, Bull. int. Acad. Praque, **33**, 38–55 (1932).
[2] C. H. Dowker: Inductive dimension of completely normal spaces, Quart. Jour. Math., **4**, 267–281 (1953).
[3] K. Morita: Normal families and dimension theory for metric spaces, Math. Ann., **128**, 350–362 (1954).
[4] ——: On the product of a normal space with a metric space, Proc. Japan Acad., **39**, 148–150 (1963).
[5] ——: Products of normal spaces with metric spaces II, to appear.
[6] K. Nagami: On the dimension of product spaces, Proc. Japan Acad., **36**, 560–563 (1960).