83. An Aspect of Local Property of $|N, p_n|$ Summability of a Factored Fourier Series

By Lal Mani TRIPATHI
Department of Mathematics, University of Allahabad, India

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1. A series $\sum a_n$ with partial sums s_n is summable to sum s by the Nörlund method (N, p_n) if

$$t_n=\left\{\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k\right\}\to s, \tag{1.1}$$

as $n\to\infty$, where $P_n=\sum_{\nu=0}^{n} p_n$ and $p_\nu>0$ [2]. The series $\sum a_n$ is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation [4]. The conditions for the regularity of the summability (N, p_n) defined by (1.1) are

$$\lim_{n\to\infty} p_n/P_n=0, \text{ and } \sum_{\nu=0}^{n}|p_\nu|=0(P_n). \tag{1.2}$$

In the special case in which

$$p_n=\binom{n+\alpha-1}{\alpha-1}=\frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \quad (\alpha>0),$$

the Nörlund mean reduces to the familiar Cesàro mean of order α [2]. And for the value for which

$$p_n=\frac{1}{n+1};\ P_n\sim\log n,$$

the Nörlund mean reduces to the harmonic mean [6].

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi,\pi)$. Without any loss of generality, we may assume that the constant term in the Fourier series of $f(t)$ is zero, that is,

$$\int_{-\pi}^{\pi} f(t)dt=0,$$

and

$$f(t)\sim\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=1}^{\infty}A_n(t).$$

We use the following notations:—

$$\phi(t)=\frac{1}{2}\left\{f(x+t)+f(x-t)-2f(x)\right\},$$

$$\Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)du \quad (\alpha>0),$$

$$\Phi_0(t)=\phi(t),$$

$$\phi_\alpha(t)=\Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t) \quad (0\leq\alpha\leq 1).$$

2. In 1957 Prasad and Bhatt [5] established the following theorem:

THEOREM A. If $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, and $\phi_\alpha(t)$ $(0\le\alpha\le 1)$ is of bounded variation in $(0,\pi)$, then the series $\sum\lambda_n A_n(t)$, at $t=x$, is summable $|C,\alpha|$.

Since a Lebesgue integral is absolutely continuous, it is plain that $\phi_1(t)$ is of bounded variation in any range (δ,π), $\delta>0$. A necessary consequence of the above result is the following theorem:

THEOREM B. The summability $|C,1|$ of the factored Fourier series $\sum\lambda_n A_n(t)$ at a given point depends only upon the behaviour of the generating function in the immediate neighbourhood of the point and is thus a local property.

Very recently Lal [3] proved the following theorem:

THEOREM C. If $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $\left|N,\dfrac{1}{n+1}\right|$ of the series $\sum\{A_n(t)\times \log(n+1)\lambda_n/n\}$ at a point can be ensured by a local property.

Applying the absolute Nörlund summability method, which is more general than both the $|C,1|$ summability and absolute harmonic summability, the object of this paper is to investigate a suitable type of factor so that the summability $|N,p_n|$ of the factored Fourier series becomes a local property.

In what follows we establish the following

THEOREM. *If $\{p_n\}$ and $\{p_n-p_{n+1}\}$ are both non-negative and non-increasing and $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then $|N,p_n|$ summability of $\sum A_n(t)\lambda_n P_n/n$ depends only on the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the point $t=x$.*

It is evident that Theorems B and C follow as special cases of our theorem in the cases in which $p_n=1$ and $p_n=\dfrac{1}{n+1}$ respectively.

3. The proof of the theorem is based on the following lemma.

Lemma ([1], Theorem 1). *Suppose that $f_n(x)$ is measurable in (a,b) where $b-a\le\infty$, for $n=1,2,\cdots$. Then a necessary and sufficient condition that, for every function $\lambda(x)$ integrable (L) over (a,b), the functions $f_n(x)\lambda(x)$ should be integrable (L) over (a,b) and*

$$\sum_{n=1}^{\infty}\left|\int_a^b \lambda(x)f_n(x)\,dx\right|\le K$$

is that

$$\sum_{n=1}^{\infty}|f_n(x)|\le K,$$

where K is an absolute constant, for almost every x in (a,b).

4. PROOF OF THEOREM. Since

$$t_n=\frac{1}{P_n}\sum_{\nu=0}^{n}P_{n-\nu}u_\nu=\frac{1}{P_n}\sum_{\nu=0}^{n}P_\nu u_{n-\nu},$$

where

$$u_n = \frac{A_n(t)P_n\lambda_n}{n},$$

we have

$$t_n - t_{n-1} = \sum_{\nu=0}^{n-1}\left(\frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}}\right)u_{n-\nu}$$
$$= \frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}(P_np_\nu - P_\nu p_n)u_{n-\nu}$$
$$= \frac{1}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}(P_np_{n-\nu-1} - P_{n-\nu-1}p_n)u_{\nu+1}.$$

For the Fourier series of $f(t)$ at $t=x$,

$$A_n(t) = \frac{2}{\pi}\int_0^\pi \phi(t)\cos nt\,dt,$$

so that

$$t_n - t_{n-1} = \frac{2}{\pi}\int_0^\pi \phi(t)\left\{\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-1}(P_np_{n-k-1} - P_{n-k-1}p_n)\frac{P_{k+1}\lambda_{k+1}}{k+1}\cos(k+1)t\right\}dt$$
$$= \frac{2}{\pi}\int_0^\pi \phi(t)K(n,t)dt,$$

say.

Hence

$$\sum_{n=2}^\infty \left|t_n - t_{n-1}\right| \le \sum_{n=2}^\infty \left|\frac{2}{\pi}\int_\delta^\pi \phi(t)K(n,t)dt\right| + \sum_{n=2}^\infty \left|\frac{2}{\pi}\int_0^\delta \phi(t)K(n,t)dt\right|.$$

Thus in order to prove the theorem we have to establish that

$$\sum_{n=2}^\infty \left|\frac{2}{\pi}\int_\delta^\pi \phi(t)K(n,t)dt\right| < \infty.$$

But by virtue of the Lemma, it is sufficient for our purpose to show that

$$(4.1) \qquad \sum_{n=2}^\infty |K(n,t)| \le A,$$

for $0<\delta\le t\le\pi$, where A is a positive constant not necessarily the same one each time it occurs.

Now

$$\sum_{n=2}^m \left|K(n,t)\right| = \sum_{n=2}^m \frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_{n-k-1} - P_{n-k-1}p_n)\frac{P_{k+1}\lambda_{k+1}\cos(k+1)t}{k+1}\right|$$
$$= \sum_{n=2}^m \frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}M(n,t)\right|,$$

say.

Applying Abel's transformation, we get

$$\sum_{k=0}^{n-1}M(n,t) = \sum_{k=0}^{n-2}\left[\varDelta\left\{(P_nP_{n-k-1} - P_{n-k-1}p_n)\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\sum_{\nu=0}^{k}\cos(\nu+1)t\right]$$
$$+ (P_np_0 - P_0p_n)\frac{P_n\lambda_n}{n}\sum_{\nu=0}^{n-1}\cos(\nu+1)t.$$

Therefore, for $0<\delta\leq t\leq\pi$, we have

$$\left|\sum_{k=0}^{n-1}M(n,t)\right|\leq A\sum_{k=0}^{n-2}\left|\Delta\left\{(P_np_{n-k-1}-P_{n-k-1}p_n)\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|+AP_n^2\frac{\lambda_n}{n}$$
$$=A[\textstyle\sum_1+\sum_2],$$

say.

Clearly

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\left|\textstyle\sum_2\right|\leq A\sum_{n=2}^{m}\lambda_n/n=O(1), \tag{4.2}$$

as $m\to\infty$.

Now

$$\textstyle\sum_1\leq\displaystyle\sum_{k=0}^{n-2}\left|\Delta\{P_np_{n-k-1}-P_{n-k-1}p_n\}\right|\left|\frac{P_{k+1}\lambda_{k+1}}{k+1}\right.$$
$$+\sum_{k=0}^{n-2}\left|(P_np_{n-k-2}-P_{n-k-2}p_n)\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|$$
$$=\textstyle\sum_{11}+\sum_{12}, \tag{4.3}$$

say.

Now

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\textstyle\sum_{11}=\displaystyle\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\sum_{k=0}^{n-2}\left|\Delta\{P_np_{n-k-1}-P_{n-k-1}p_n\}\right|\left|\frac{P_{k+1}\lambda_{k+1}}{k+1}\right.$$
$$=\sum_{k=0}^{m-2}\frac{P_{k+1}\lambda_{k+1}}{k+1}\sum_{n=k+2}^{m}\frac{|\Delta(P_np_{n-k-1}-P_{n-k-1}p_n)|}{P_nP_{n-1}}$$
$$\leq\sum_{k=0}^{m-2}\left(\frac{P_{k+1}\lambda_{k+1}}{k+1}\right)\sum_{n=k+2}^{m}\left[\frac{|\Delta p_{n-k-1}|}{P_{n-1}}+\frac{p_{n-k-1}p_n}{P_nP_{n-1}}\right]$$
$$=O\left[\sum_{k=0}^{m-2}\frac{\lambda_{k+1}}{k+1}\sum_{n=k+2}^{m}|\Delta p_{n-k-1}|\right]$$
$$+O\left[\sum_{k=0}^{m-2}\frac{P_{k+1}\lambda_{k+1}}{k+1}\sum_{n=k+2}^{m}\frac{p_n}{P_nP_{n-1}}\right]$$
$$=O\left[\sum_{k=0}^{m-2}\frac{\lambda_{k+1}}{k+1}\right] \tag{4.4}$$
$$=O(1).$$

Again

$$\textstyle\sum_{12}=\displaystyle\sum_{k=0}^{n-2}\left|(P_np_{n-k-2}-P_{n-k-2}p_n)\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|$$
$$\leq\sum_{k=0}^{n-2}(P_n-P_{n-k-2})p_n\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|$$
$$+\sum_{k=0}^{n-2}(p_{n-k-2}-p_n)P_n\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|$$
$$=\textstyle\sum_{121}+\sum_{122}, \tag{4.5}$$

say.

Now

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\textstyle\sum_{121}=\displaystyle\sum_{n=2}^{m}\frac{p_n}{P_nP_{n-1}}\sum_{k=0}^{n-2}(P_n-P_{n-k-2})\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|$$

$$=\sum_{k=0}^{m-2}\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|\sum_{n=k+2}^{m}(P_n-P_{n-k-2})\Delta\left(\frac{1}{P_{n-1}}\right)$$

$$\leq A\sum_{k=0}^{m-2}\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|P_{k+2}\sum_{n=k+2}^{m}\Delta\left(\frac{1}{P_{n-1}}\right)$$

$$=O\left[\sum_{k=0}^{m-2}\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|\right]$$

$$=O\left[\sum_{k=0}^{m-2}\left|\frac{p_{k+1}\lambda_{k+1}}{k+1}\right]+O\left[\sum_{k=0}^{m-2}\frac{P_{k+1}\Delta\lambda_{k+1}}{k+1}\right]$$

$$+O\left[\sum_{k=0}^{m-2}\frac{\lambda_{k+1}}{k+1}\,\frac{P_{k+1}}{k+2}\right]$$

$$=O\left[\sum_{k=0}^{m-2}\frac{\lambda_{k+1}}{k+1}\right]+O\left[\sum_{k=0}^{m-2}\Delta\lambda_{k+1}\right]$$

$$=O(1), \tag{4.6}$$

as $m\to\infty$, since P_n-P_{n-k-2} decreases as n increases.

Also

$$\sum_{n=2}^{m}\frac{1}{P_nP_{n-1}}\sum\nolimits_{122}=\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2}-p_n)\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|$$

$$\leq\sum_{n=2}^{m}\frac{1}{P_{n-1}}\sum_{k=0}^{n-2}(p_{n-k-2}-p_{n-k-1})(k+2)\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|$$

$$=\sum_{k=0}^{m-2}(k+2)\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|\sum_{n=k+2}^{m}\frac{p_{n-k-2}-p_{n-k-1}}{P_{n-1}}$$

$$\leq\sum_{k=0}^{m-2}\frac{(k+2)}{P_{k+1}}\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|\sum_{n=k+2}^{m}(p_{n-k-2}-p_{n-k-1})$$

$$=O\left[\sum_{k=0}^{m-2}\frac{(k+2)}{P_{k+1}}\left|\Delta\left\{\frac{P_{k+1}\lambda_{k+1}}{k+1}\right\}\right|\right]$$

$$=O\left[\sum_{k=0}^{m-2}\frac{(k+2)p_{k+1}}{P_{k+1}}\,\frac{\lambda_{k+1}}{k+1}\right]+O\left[\sum_{k=0}^{m-2}\Delta\lambda_{k+1}\right]$$

$$+O\left[\sum_{k=0}^{m-2}\frac{\lambda_{k+1}}{k+1}\right]$$

$$=O\left[\sum_{k=0}^{m-2}\frac{\lambda_{k+1}}{k+1}\right]+O\left[\sum_{k=0}^{m-2}\Delta\lambda_{k+1}\right]$$

$$=O(1). \tag{4.7}$$

With the help of results from (4.2) to (4.7), (4.1) follows, which completes the proof of the theorem.

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