132. *The Area of Nonparametric Measurable Surfaces*

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1. **Basic notions.** We shall understand by a *rectangle* any closed nondegenerate interval of the Euclidean plane $\boldsymbol{R}^2$. The letter I will be reserved to denote a rectangle. Let $I=[a_1, b_1; a_2, b_2]$ explicitly. When $0<\alpha<1$ and $2\alpha<\min(b_1-a_1, b_2-a_2)$, we say that α is *admissible* for I and we find it convenient to write

$$I_\alpha=[a_1+\alpha, b_1-\alpha; a_2+\alpha, b_2-\alpha].$$

Further, $\mathrm{Rec}\, I$ will denote the class of all subrectangles of I (inclusive of I itself).

Suppose that T is an *additive continuous map* of $\mathrm{Rec}\, I$ into the Euclidean space $\boldsymbol{R}^m$ of dimension m. In other words, let the m coordinates of the point $T(J)$, where $J\in\mathrm{Rec}\, I$, be additive continuous functions of J in the usual sense [Saks 4, Chap. III]. If α is any admissible number for I, the quotient

$$T_\alpha(x, y)=T([x-\alpha, x+\alpha; y-\alpha, y+\alpha])/(4\alpha^2),$$

defined for the points $\langle x, y\rangle$ of the rectangle I_α, is obviously a continuous map of I_α into the space $\boldsymbol{R}^m$. We may say that T_α is the *squarewise mean* of T (for squares of side-length 2α).

Let g denote generically a continuous map of a rectangle K into $\boldsymbol{R}^m$, and let Ψ be a functional which assigns to each g a nonnegative value $\Psi(g)=\Psi(g; K)\leqq+\infty$. (It should be noted that not only the map g, but also the rectangle K is supposed arbitrary; the space $\boldsymbol{R}^m$, however, is kept fixed.) If J is a subrectangle of K, the partial map $g|J$ is continuous on J and we shall write $\Psi(g; J)$ for $\Psi(g|J)$.

Given as above the map T and the functional Ψ, let t be a generic continuous map of I into $\boldsymbol{R}^m$. We shall denote by $M(\Psi, T)$, or more expressly $M(\Psi, T; I)$, the lower limit of $\Psi(t; I_\alpha)$ as $\alpha\to 0$ and $\rho(T_\alpha, t; I_\alpha)\to 0$ simultaneously, where $\alpha, I_\alpha, T_\alpha$ have the aforesaid meaning and ρ indicates the ordinary distance, on I_α, between the two maps T_α and t. In other words, $M(\Psi, T)$ means the supremum of $M(\beta, \Psi, T)$ for all $\beta>0$, where $M(\beta, \Psi, T)$ is the infimum of $\Psi(t; I_\alpha)$ for all pairs $\langle\alpha, t\rangle$ such that $\alpha<\beta$ and $\rho(T_\alpha, t; I_\alpha)<\beta$. (The last inequality is fulfilled if, for example, we choose for t any continuous extension of T_α to the whole rectangle I.)

2. **Aim of the note.** By a *nonparametric measurable surface* we shall mean a surface of the form $z=f(x, y)$, where f is a finite measurable function on a rectangle. We are interested in the theory

of area of such a surface. As far as the author knows, Cesari [1] was the first to give a successful definition of the area in question. In the present note we shall introduce another definition of area, by specializing suitably the quantity $M(\Psi, T)$ obtained just now. Our main result will assert that the new area coincides identically with Cesari's.

The process of specialization of $M(\Psi, T)$ might find further applications to the geometry of surfaces. In particular, we could thus generalize the well-known notion of "integral curvature" of smooth surfaces to the case of nonparametric continuous surfaces on a rectangle. Space limitation prevents us, however, from touching upon this subject.

3. Area of nonparametric measurable surfaces. Let f be a finite measurable function on I. (A *function*, by itself, will exclusively mean a real-valued one.) An additive continuous rectangle-function F defined on the class Rec I will be termed a *primitive* for f, whenever at almost every point $\langle x, y\rangle$ of I the ordinary derivative $F'(x, y)$ exists and coincides with $f(x, y)$. By Lusin's theorem [Saks 4, p. 218, small print], there always exist such functions F.

If g is an arbitrary continuous function on a rectangle K, the Lebesgue area of the nonparametric surface $z=g(x, y)$ will be written $S(g)$ or $S(g; K)$, as in [Saks 4, Chap. V].

Returning to the function f, let F be any primitive for f. In the terminology of §1, the function F is an additive continuous map of Rec I into the real line $\boldsymbol{R}$. We now specialize the functional Ψ of the same section to be the Lebesgue area S just considered, i.e. we put $\Psi(g; K)=S(g; K)$ identically. The quantity $M(S, F; I)$ is then well determined.

We are now in a position to define the area, $A(f)$ or $A(f; I)$, of the nonparametric measurable surface $z=f(x, y)$. Namely we set $A(f)$ to be the infimum of $M(S, F; I)$ for all choices of the primitive function F. If J is any subrectangle of I, we shall write $A(f; J)$ for $A(f\,|\,J)$.

4. Area of nonparametric summable surfaces. In this section we suppose f to be a finite summable function on I. For each rectangle $J\subset I$ let $F(J)$ denote the integral of f over J. Since F is then a primitive for f by a standard theorem, the consideration of the preceding § applies and yields the quantity $M(S, F; I)$.

On the other hand, we defined in our previous note [3] an area of the nonparametric summable surface $z=f(x, y)$ and denoted it by $L(f)$ or $L(f; I)$. We readily find, however, that *this area* $L(f)$ *is no other than* $M(S, F; I)$, *so that* $A(f)\leqq L(f)$. Indeed, if α is admissible for I and $\langle x, y\rangle$ is any point of I_α,

$$F_\alpha(x, y)=\frac{1}{4\alpha^2}F([x, y; \alpha])=\frac{1}{4\alpha^2}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}f(x+u, y+v)dudv,$$

where $[x, y; \alpha]$ abbreviates $[x-\alpha, x+\alpha; y-\alpha, y+\alpha]$. Thus the squarewise mean F_α is the integral mean of the function f.

5. The Cesari area. We return to the situation of § 3 and suppose again that f is a finite measurable function on I. In his paper [1], Cesari introduced the area $L^*(f)$ of the surface $z=f(x, y)$ as follows. Let $\xi=\langle p_n; n=1, 2, \cdots\rangle$ be an arbitrary infinite sequence of polyhedral functions on I, such that for almost every point $\langle x, y\rangle$ of I, we have $p_n(x, y)\to f(x, y)$ as $n\to+\infty$. It may be observed that the existence of ξ is an immediate consequence of Lusin's theorem [Saks 4, p. 72] and Tietze's extension theorem. We define now

$$L^*(f)=L^*(f; I)=\operatorname*{Inf}_{\xi}\,[\liminf_{n} S(p_n; I)],$$

where S denotes the Lebesgue area as before. (We write Inf for "infimum" and inf for "inferior", in order to avoid any ambiguity.)

Remarks. (1) We could have used continuous, instead of polyhedral, functions in the foregoing definition. The verification is immediate.

(2) $L^*(f; J)$ has an obvious meaning for $J\in \operatorname{Rec} I$ and is evidently a monotone nondecreasing function of J.

6. Identification of the Cesari area with ours. As above, suppose f a finite measurable function on I. We are concerned with proving the identity $L^*(f)=A(f)$, and we begin with the following

Lemma (i). $L^*(f)\leqq A(f)$.

Proof. It suffices to show that $L^*(f; I)\leqq M(S, F; I)$ whenever F is a primitive for the function f. To each positive integer n we can, by the definition of $M(S, F; I)$, make correspond a number $\alpha(n)$ admissible for I and a continuous function g_n on I, such that $\alpha(n)<n^{-1}$, $\rho(F_{\alpha(n)}, g_n; I_{\alpha(n)})<n^{-1}$, and

$$S(g_n; I_{\alpha(n)})\leqq M(S, F; I)+n^{-1}.$$

(Of course $F_{\alpha(n)}$ denotes the squarewise mean of F, and $I_{\alpha(n)}$ the rectangle on which $F_{\alpha(n)}$ is defined, in accordance with § 1. On the other hand, since n is an integer and so not admissible for I, the use of n as subscript in g_n cannot give rise to any confusion.)

It follows at once that $\lim g_n(x, y)=f(x, y)$ almost everywhere on I. Hence, if J is any rectangle fixed in the interior of I, the remarks of § 5 and our last inequality together yield

$$L^*(f; J)\leqq \liminf S(g_n; J)\leqq \liminf S(g_n; I_{\alpha(n)})\leqq M(S, F; I).$$

The lemma will therefore be established as soon as we have ascertained the following assertion.

Lemma (ii). *$L^*(f; I)$ is the supremum of $L^*(f; J)$ for all rectangles J situated in the interior of I.*

PROOF. For any positive number r and any set X in the plane, we shall write $rX=\{rw; w\in X\}$. Further, if φ is a finite function on the rectangle rI, the symbols $\varphi^{(r)}$ and $\varphi^{[r]}$ will denote the functions defined for all $w\in I$ respectively by

$$\varphi^{(r)}(w)=\varphi(rw) \quad \text{and} \quad \varphi^{[r]}(w)=r^{-1}\cdot\varphi(rw).$$

To prove the lemma, we may and do assume that the centre of I is the origin of the plane. Suppose given a positive number $\delta\leqq 1$. On account of Lusin's theorem [Saks 4, p. 72], there exists in the interior of I a compact set K for which $|I-K|<\delta^2$ and on which the function f is continuous. We fix this set K.

It is clearly possible to choose in the open interval $(1-\delta, 1)$ a number c so as to fulfil the three conditions $c^{-1}K\subset I$,

$$\rho(f^{(c)}, f^{[c]}; c^{-1}K)<\delta/2, \quad \text{and} \quad \rho(f, f^{(c)}; K')<\delta/2,$$

where and subsequently we write for short $K'=K\frown c^{-1}K$. (Indeed any $c<1$ sufficiently near 1 will serve the purpose.) We must then have the inequality

$$(*) \qquad \rho(f, f^{[c]}; K')<\delta.$$

The definition of the Cesari area $L^*(f; cI)$ implies the existence of an infinite sequence of polyhedral functions $p_1, p_2, \cdots$ defined on cI, such that $\lim p_i(w)=f(w)$ for almost every $w\in cI$ and that $\liminf S(p_i; cI)\leqq L^*(f; cI)+c^2\delta/2$. Applying Egorov's theorem [Saks 4, p. 18] to this sequence $\langle p_i\rangle$, we can find in cI a measurable set E for which $|cI-E|<c^2\delta^2$ and on which $\langle p_i\rangle$ converges uniformly to f. It follows at once that there exists on cI a polyhedral function p which satisfies both

$$\rho(p, f; E)<c\delta \quad \text{and} \quad S(p; cI)<L^*(f; cI)+c^2\delta.$$

Consider the function $p^{[c]}$, which is plainly polyhedral on I. We verify immediately $\rho(p^{[c]}, f^{[c]}; c^{-1}E)=c^{-1}\cdot\rho(p, f; E)<\delta$ and

$$S(p^{[c]}; I)=c^{-2}\cdot S(p; cI)<c^{-2}\cdot L^*(f; cI)+\delta.$$

In view of the inequality $(*)$ we find further $\rho(p^{[c]}, f; K^*)<2\delta$, where $K^*=K'\frown c^{-1}E$. This set K^* fulfils moreover

$$|I-K^*|\leqq|I-K|+|I-c^{-1}K|+|I-c^{-1}E|<\delta^2+\delta^2+\delta^2=3\delta^2.$$

Let $n=1, 2, \cdots$ and let us specialize $\delta=n^{-1}$ in the foregoing. The number c, the function $p^{[c]}$, and the set K^* now depend on n, and we shall denote them by c_n, q_n, K_n respectively. Summarizing what has already been established, we find that $1-n^{-1}<c_n<1$, that q_n is a polyhedral function on I, that K_n is a measurable subset of I for which $|I-K_n|<3/n^2$, and that

$$\rho(q_n, f; K_n)<2/n, \qquad S(q_n; I)<c_n^{-2}\cdot L^*(f; c_nI)+n^{-1}.$$

Define finally $H_n=K_n\frown K_{n+1}\frown\cdots$ for $n=1, 2, \cdots$. Then $H_1, H_2, \cdots$ form an ascending sequence of measurable subsets of I and we have $|I-H_n|\leqq|I-K_n|+|I-K_{n+1}|+\cdots<6/n$ for every n. Accordingly, if we write $H=\lim H_n=\liminf K_n$, we get $|I-H|=0$. Furthermore, each point w of H belongs to all but a finite number of the sets K_n, so that $|q_n(w)-f(w)|<2/n$ for n sufficiently large. The sequence $\langle q_n\rangle$ thus converges to the function f almost everywhere on I. It follows by the definition of $L^*(f;I)$ and the above estimation of $S(q_n;I)$ that

$$L^*(f;I)\leqq \liminf S(q_n;I)\leqq \liminf c_n^{-2}\cdot L^*(f;c_nI).$$

If, therefore, σ denotes the supremum appearing in our lemma, then $L^*(f;I)\leqq \liminf \sigma/c_n^2=\sigma$. This completes the proof since evidently $L^*(f;I)\geqq\sigma$.

LEMMA (iii). *If g is a continuously differentiable function on a rectangle $I=[a, a+h; b, b+k]$, there always exists in I at least one point $\langle x_0, y_0\rangle$ with the following property*:

$$(I)\iint|g(x,y)-g(x_0,y_0)|\,dxdy\leqq 2\cdot(I)\iint\left(h\left|\frac{\partial g}{\partial x}\right|+k\left|\frac{\partial g}{\partial y}\right|\right)dxdy.$$

The proof is left to the reader.

LEMMA (iv). *The finiteness of the area $L^*(f)$ implies the summability over I of the function f.*

PROOF. By hypothesis there is on I a sequence of polyhedral functions $\langle p_n; n=1,2,\cdots\rangle$ tending almost everywhere on I to f and such that the corresponding areas $S(p_n)$ form a bounded sequence. Let us associate with each n a continuously differentiable function g_n on I subject to the conditions

$$\rho(p_n, g_n; I)<n^{-1} \quad\text{and}\quad S(g_n)<S(p_n)+n^{-1},$$

the existence of such a function being ensured by Radò's theorem [Saks 4, p. 179]. Then $g_n\to f$ almost everywhere on I as $n\to+\infty$, and the sequence $\langle S(g_n)\rangle$ is bounded. The supremum of this sequence will be written σ, so that $\sigma<+\infty$.

Now $S(g_n)$ is given, for each n, by the well-known formula

$$S(g_n)=(I)\iint\sqrt{\left(\frac{\partial g_n}{\partial x}\right)^2+\left(\frac{\partial g_n}{\partial y}\right)^2+1}\;dxdy.$$

This, combined with $S(g_n)\leqq\sigma$, shows that

$$(I)\iint\left|\frac{\partial g_n}{\partial x}\right|dxdy\leqq\sigma \quad\text{and}\quad (I)\iint\left|\frac{\partial g_n}{\partial y}\right|dxdy\leqq\sigma.$$

If, therefore, we put explicitly $I=[a, a+h; b, b+k]$, it follows at once by lemma (iii) that for each $n=1,2,\cdots$ there exists in I a point $\langle x_n, y_n\rangle$ fulfilling the inequality

$$(I)\iint|g_n(x,y)-g_n(x_n,y_n)|\,dxdy\leqq 2(h+k)\sigma.$$

Henceforth we shall write for simplicity $\gamma_n=g_n(x_n, y_n)$.

Consider the sequence $\gamma_1, \gamma_2, \cdots$. Without losing generality we may assume the existence of the limit γ (finite or $\pm\infty$) of this sequence; for otherwise we need only pass to a suitable subsequence. Since $\lim g_n(x,y)=f(x,y)$ almost everywhere on I (see above), the function $g_n(x, y)-\gamma_n$ tends, as $n\to+\infty$, to $f(x, y)-\gamma$ almost everywhere on I. (It should be remarked that if $\gamma=\pm\infty$, then $\gamma-\gamma$ is understood to vanish.) In consequence, by Fatou's lemma and what has already been stated, we obtain

$$(I)\iint|f(x, y)-\gamma|\,dxdy\leqq\liminf_n\,(I)\iint|g_n(x, y)-\gamma_n|\,dxdy$$
$$\leqq 2(h+k)\sigma.$$

From this we deduce successively that γ is finite and that f is summable on I. The lemma is thus established.

THEOREM. *We have $L^*(f)=A(f)$ for every finite measurable function f on a rectangle I.*

PROOF. In virtue of lemma (i) it suffices to derive the relation $A(f)\leqq L^*(f)$ in the case in which $L^*(f)<+\infty$. Then f must be summable over I by lemma (iv). The Goffman area $\Phi(f)$ of the surface $z=f(x, y)$ therefore exists [Goffman 2], and we have $\Phi(f)=L^*(f)$ by theorem 5 of [2]. On the other hand we have already seen in §4 that $A(f)$ cannot exceed the area $L(f)$ of our note [3]. But it was proved in [3, §5] that the two functionals Φ and L coincide identically. Hence the desired inequality follows, and the proof is complete.

REMARK. The preceding proof leans heavily on the two identities $L^*(\varphi)=\Phi(\varphi)$ and $\Phi(\varphi)=L(\varphi)$, or more accurately, on their consequence $L^*(\varphi)=L(\varphi)$, where φ is a finite summable function on a rectangle. Now, as we recall, $\Phi(\varphi)=L(\varphi)$ was established in [3] without having recourse to [1] or [2]. There springs up the question whether it be possible to derive independently also $L^*(\varphi)=L(\varphi)$, without intermediation of the Goffman area. We are unable to settle this at present. (Needless to say, it is only the inequality $L(\varphi)\leqq L^*(\varphi)$ that matters.)

References

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