16. The Causality Condition in Nowhere Dense Perfect Model

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§1. Introduction. From the research about the source (satisfying the causality condition) which is effective to cut-off, nowhere dense perfect model (or A-integral model) was found in [4]. But even if this model be used, it seems that the usual causality condition is not necessarily satisfied. In §2 the causality condition rewritten to the mathematical form by using a sort of modeling is investigated, and the milder causality condition (found from this modeling) which is satisfied by the nowhere dense perfect set is looked for. Furthermore the fitness of this causality condition for the various examples is investigated. Next the fitness of this condition for two and three dimensional nowhere dense perfect model is also investigated. The coincidence between this model of causality condition and the usual causality condition is not necessarily obvious. But it will be sure that there is some relation between them, and it seems valuable that Dini's derivative corresponding to the finite difference for non-local field theory can be obtained in fully exact form.

In §3 A-integral representation of distribution whose carrier satisfies the above mild causality condition is constructed. Furthermore it is stated that the sequence of A-integral representations whose carrier satisfies the global causality condition can be also constructed. In §4 the various criterions of this research are shown.

§2. The discriptions of causality condition by using a sort of modeling form.

Consider the two dimensional Euclid space with coordinate (t,x), and one dimensional set E defined on the x axis. Next, construct the following function $t=f(x)=\int_0^x \varphi(x)\,dx$ by using the function $\varphi(x)=\begin{cases} 1/c & \text{for } x\in E^c \text{ (the complement set of } E) \\ 0 & \text{for } x\in E, \end{cases}$

where c is a constant corresponding to the light velocity. This function $t=f(x)$ can be rewritten to the form $x=f^{-1}(t)$ (not necessarily one valued with respect to t).

Definition 1. If $0<(x_1-x_2)/(f(x_1)-f(x_2))\leqq c$ holds good for any pair (x_1, x_2) with the property $x_1\neq x_2$, this set E is called global causal set.

Definition 2. If the function $x=f^{-1}(t)$ is one valued, and if all possible usual differential coefficients of this function are not larger than c, then this set E is called local causal set.

Now, by using Dini's lower derivate $\underline{D}f^{-1}(t)$, generalized causal set can be defined.

Definition 3. If the function $x=f^{-1}(t)$ is one valued, if $\underline{D}f^{-1}(t)$ can be defined for all t, and if it is equal to c, this set E is called generalized causal set.

The local causal set treated in this article is also generalized causal set. Furthermore $\int \underline{D}f^{-1}(t)dt$ becomes to $ct+d$ and this set E is considered as the global causal set with respect to this Dini's lower derivate.

Here, for various considerable sets, it is investigated whether the conditions in Definitions 1–3 (which is used for only the classification of the type of the set) are satisfied or not.

Example 1. The set consisting of finite points becomes to a global, generalized and local causal set.

Example 2. An interval (open, closed or others) is not a global, generalized and local causal set.

Since the properties of the above sets are well known, the facts described in Examples 1–2 are obvious.

Let E_1 denote the nowhere dense perfect set constructed in [4] with positive measure contained in the interval [0, 1].

Lemma 1. *E_1 is not a global causal set but a local causal set. Furthermore it is also a generalized causal set.*

Proof. Since the measure of E_1 is positive, it follows that $(1-0)/(f(+1)-f(0))=1/\{(1\text{-mes } E_1)/c\}=c/(1\text{-mes } E_1)>c$ holds good. Then E_1 is not a global causal set. At the each point in the open complementary set E_1^c, it is obvious that $1/f'(x)=c$ holds good. Next choose an arbitrary point x_0 contained in E_1. From the definition of nowhere dense perfect set it is the end point of the closure of an open connected component contained in E_1^c. If it is the left end of this component, $1/(D_{\oplus}f(x_0))=c$, and if it is the right end of it, $1/(D_{\ominus}f(x_0))=c$.

If $1/(D_{\oplus}f(x_0))=c$, then $1/(\bar{D}_{\ominus}f(x_0))=(3/2)\,c\geq c$, and

if $1/(D_{\ominus}f(x_0))=c$, then $1/(\bar{D}_{\oplus}f(x_0))=(3/2)c\geq c$.

Hence in each point of E_1, the usual differential coefficient cannot be obtained. Furthermore, Dini's lower derivate can be defined and is equal to c.

Then E_1 is the local and generalized causal set.

From the same arguments as one in Lemma 1 the following result is obtained.

Example 3. Nowhere dense perfect set with positive measure is not a global causal set but a local and generalized causal set.

For $f(x)$ related to the global causal set E, the usual differential coefficient $Df(x)$ can be defined for all x. Then we assert the following.

Theorem 1. *The family of the global causal sets is contained in the family of the generalized causal sets. And the family of the generalized causal sets is contained in the family of the local causal sets.*

Let l denote a straight line, and E denote a two or three dimensional set.

Definition 4. If $l \cap E$ is the sum of a nowhere dense perfect set (may be void set) and an isolated points set (may be void set) for any l, then we say that E is nowhere dense to any direction.

The same arguments with respect to a fixed direction are also possible. Let $\widetilde{E}_i$ $(i=1, 2, 3)$ denote the nowhere dense perfect sets.

Example 4. $\widetilde{E}_1 \times \widetilde{E}_2 \times \widetilde{E}_3$ is nowhere dense to any direction. Let I_i $(i=1, 2, 3)$ denote the closed intervals.

Example 5. $\widetilde{E}_1 \times I_2 \times I_3$ (or $\widetilde{E}_1 \times \widetilde{E}_2 \times I_3$) is nowhere dense to fixed direction not to be pararell to the plane $I_2 \times I_3$ (I_3).

The spherical symmetric nowhere dense perfect set in E^3 (which is nowhere dense to any direction) can be also constructed.

§3. The countable infinite sum of nowhere dense perfect sets. In this paragraph, examples of the local causal dense set, the generalized causal dense set and the global causal dense set are shown. The most important one of them is the local and generalized causal dense set which is constructed by the sum of the countable infinite nowhere dense perfect sets with positive measure. Furthermore A-integral representation of distribution (by Бонди [2] and Виноградова [1]) defined on this set is constructed.

Example 6. The dense set E consisting of all rational points in real axis is the global causal dense set with total measure 0.

Example 7. Let E_1 denote the nowhere dense perfect set shown in [4] as example which is the subset of the interval $[0, 1]$, and E_n denote the $1/n$ similar reduction of E_1. At the first step, E_1 is arranged in the interval $[2n, 2n+1]$ (for $n=0, \pm 1, \pm 2, \cdots$) and this arranged set is denoted by F_1. At the second step, $E_{2\times 4}$ is arranged in the middle of the open connected component of F_1^C whose length is larger than $3/2$, and this arranged set is denoted by F_2. Iterating the same processes (countable infinite times) the set F_∞ is constructed. This F_∞ is the above local and generalized causal dense set.

Let's select an arbitrary nowhere dense perfect set defined in an open finite interval (a, b) and denote it by $E(a, b)$.

Lemma 2. *For any positive number $\varepsilon>0$, there exists a positive integer N such that the distance between the two neighbouring open connected elements of $(a, b)-E(a, b)$ whose length is larger than $1/N$ is always smaller than ε.*

Proof. Suppose that the finite open covering of closed interval $[a, b]$ chosen from the family of the $\varepsilon/2$ neighbourhoods $U_{\varepsilon/2}(x_i)$ of all rational points x_i consists of n neighbourhoods $U_{\varepsilon/2}(\tilde{x}_j)$ $(j=1, 2, \cdots, n)$. Since E is nowhere dense, each neighbourhood contains the point $\tilde{y}_j$ in some open connected component of the set $(a, b)-E(a, b)$. We denote by l_j $(j=1, 2, \cdots, n)$ the length of the above open connected component which contains $\tilde{y}_j$. The positive integer N with the property $l_j<1/N (j=1, 2, \cdots, n)$ is the required N.

From this Lemma 3 it is deduced that F_∞ has the following character.

(1) For positive number $\varepsilon>0$ with the property $3/2^n>\varepsilon/3$, there exists a fixed positive integer k such that the distance between any two neighbouring nowhere dense perfect sets $(E_{2^l\times 4^l}, n\leq l\leq n+k-1)$ used for the construction of $F_{n+k}-F_n$ is smaller than $\varepsilon>0$.

Lemma 3. *F_∞ is a local and generalized causal set.*

Proof. For any point p (contained in F_∞), there exists a positive integer n (depending on p) such that p is contained in F_n. Since F_n is a nowhere dense perfect set, p must be the boundary point of an open connected component $I_{n,k}$ of F_n^C. According to the argument in Lemma 1, F_n is a generalized and local causal set. Therefore it is only remained to consider the effect of the set $F_\infty-F_n$. The sequence $\{x_m\}$ constructed by the elements in $I_{n,k}$ with the property dist $(p, x_m)=o(1/2^m)$ can be chosen so as to satisfy the condition $\left|\int_p^{x_m} I_{F_\infty-F_n}(x)\,dx\right|=o(1/4^m)$, where $I_{F_\infty-F_n}(x)$ is the characteristic function of $F_\infty-F_n$. As m tends to ∞, $\left|\int_p^{x_m} I_{F_\infty-F_n}(x)\,dx\right|/\text{dist }(p, x_m)=o(1/2^m)$ tends to zero. From the above result it can be easily seen that $F_\infty-F_n$ does not give the effect to the requirement of local and generalized causality for the point p. Then the following conclusion of this Lemma is asserted.

(2) F_∞ is the local and generalized causal set.

Definition 5. If the function $f(x)$ defined in the interval $[a, b]$ satisfy the following two conditions.

1) mes $\{x;\ x\in[a, b],\ |f(x)|>n\}=o(1/n)$

2) there exists a limit $\lim_{n\to\infty}\int_a^b [f]_n(x)\,dx$,

then we say that $f(x)$ is A-integrable and the above limit is A-integral of $f(x)$ [1–3].

Theorem 2. *On this F_∞, A-integral representation in [1–2] of an arbitrary distribution can be constructed.*

Proof. At the first step, the derivative of measurable function is treated. Only the proof of this step is needed for our purpose. The measurable function $F(x)$ can be represented by the series of continuous functions $F_l(x)$ such that $\delta_l = w[F_l, [0, 2]]$ (the amplitude of $F_l(x)$ in the interval $[0, 2]$) tends to zero as l tends to ∞. Namely $F(x) = \sum_{l=1}^{\infty} F_l(x)$ holds good.

This series can be chosen by using the method in [2].

Let $\{\varepsilon_l\}$ denote the decreasing sequence of positive numbers with the property $\lim_{l\to\infty} \varepsilon_l = 0$ and $\{n_l'\}$ denote the subsequence of the natural number's sequence such that the distance between any two neighbouring nowhere dense perfect sets used for the construction of $F_{n'_{l+1}} - F_{n'_l}$ is smaller than $\varepsilon_l > 0$.

From Lemma 2 the choice of the sequence $\{n_l'\}$ can be possible. Let $\{n_l\}$ denote the subsequence of the natural number's sequence with the property $\sum_{k=l}^{\infty} 1/n_k \leq 2/n_l$. For example, the subsequence of $\{[\exp n]\}$ satisfies this property.

The density function $f_l(x)$ with the following properties is constructed from the function $F_l(x)$ by using the suitable choice of $\{\varepsilon_l\}$ and $\{n_l\}$.

a) The values of $f_l(x)$ are only $\pm n_l$, and the carrier of $f_l(x)$ is the subset of $F_{n'_{l+1}} - F_{n'_l}$.

b) $f_l(x)$ which is defined in the subset of $F_{n'_{l+1}} - F_{n1'_l}$ satisfies the condition $|\sum_{i=1}^{l} F_i(x) - \sum_{i=1}^{l} \int_0^x f_i(x)\,dx\,| < \delta_l/3$.

Since δ_l tends to zero as l tends to ∞, it follows from the property $\sum_{k=l}^{\infty} 1/n_k \leq 2/n_l$ that the condition $\operatorname{mes}\{x;\ x \in [0, 2],\ |f(x)| > n_l\} = 0(1/n_l)$ holds good.

Furthermore, from the condition $|\sum_{i=1}^{l} F_i(x) - \sum_{i=1}^{l} \int_0^x f_i(x)\,dx\,| < \delta_l/3$ it follows that there exists a limit $\lim_{n\to\infty} \int_0^x [\sum_{i=1}^{\infty} f_i]_n(x)\,dx$.

Then $f(x) = \sum_{i=1}^{\infty} f_i(x)$ is A-integrable function defined in $[0, 2]$.

By using the summation, we can easily take off the restriction related to the interval $[0, 2]$, and construct the locally A-integrable function defined in $(-\infty, \infty)$.

By using the same estimation as one used for the proof in [2] this function can be represented as $F'(x)$ by the distribution's meaning.

Next by using the above conclusion and the same estimation as one used for the proof in [2], the conclusion of this Theorem 2 can be easily proved.

Using this representation of distribution defined on the local causal set, the same arguments as [4] can be iterated and the same result as [4] can be obtained.

Furthermore, by using the sequence of this representations, the representation of distribution defined on the global causal set can be also constructed.

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References

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