14. A Remark on the Uniqueness of the Non-characteristic Cauchy Problem for Equations of Parabolic Type

By Akira TSUTSUMI

University of Osaka Prefecture

(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1965)

1. We shall consider the Cauchy problem of the equations written in the following form in $[-T, T]\times D$, where D is the closure of a domain with smooth boundary ∂D, in $n+1$-dimensional euclidean space $R_x^n \times R_t^1$;

$$(1)\qquad Pu=\left(\frac{\partial^m}{\partial t^m}-\sum_{j=0}^{m-1}\sum_{j+m|\alpha:\mathfrak{m}|\leq m} a_{j,\alpha}(t, x)\frac{\partial^{j+|\alpha|}}{\partial t^j \partial x^\alpha}\right)u(t, x)=0,$$

with the null initial data;

$$(2)\qquad \frac{\partial^\gamma}{\partial t^\gamma}u(0, x)=0 \quad x\in D,\ \gamma=0, 1, \cdots, m-1,$$

the notations contained in the above mean

m integer, $(t, x)=(t, x_1, \cdots, x_n)$

$\mathfrak{m}=(m_1, \cdots, m_n)$ m_j positive integers,

$\alpha=(\alpha_1, \cdots, \alpha_n)$ α_j non-negative integers and $|\alpha:\mathfrak{m}|=\sum_{j=1}^{n}\frac{\alpha_j}{m_j}$.

On this problem H. Kumanogo [1] and the author [2] obtained some results by the method of Carleman. But the both do not give any answer for the validity of the uniqueness in a neighborhood of the point where all $a_{j,\alpha}(t, x)$ vanish. On the other hand by eleving the regularity with respect to x and restricting the growth of derivatives of $a_{j,\alpha}(t, x)$, De Giorgi [3] obtained the uniqueness for (1) (2) in the case of two independent variables. We shall obtain an answer for the above question by extending De Giorgi's result for $n+1$ independent variables. The method is essentially the same as him. Recently G. Talenti [4] proved the uniqueness and existence for (1) with a special right hand side by extending M. Pucci's result [5] for two independent variables. His uniqueness theorem is for solutions in some Gevrey class and ours for genuine solutions. Y. Ôya [6] proved the existence and uniqueness of the Cauchy problem for the weakly hyperbolic equations which contain (1) as a special case and he assumes that $a_{j,\alpha}(t, x)$ are in some Gevery class with respect to both t and x, but with respect to t we only assume they are continuous.

2. *Theorem. We assume*

1) *There exist positive constants* $A_{j,\alpha}$ *and* ρ *such that*

$$\left|\frac{\partial^{|s|}}{\partial x^s} a_{j,\alpha}(t, x)\right| \leqq A_{j,\alpha}\rho^{|s|}\Gamma(m\,|\,s : \mathfrak{m}\,|+1)$$

holds for any multi-integer s and (t, x) in $[-T, T]\times D$.[1]

2) *$u(t, x)$ has its continuous derivatives $\frac{\partial^{j+|\beta|}}{\partial t^j \partial x^\beta}u(t, x)$ for $\frac{j}{m}+|\beta : \mathfrak{m}|\leq 1$, and satisfies* (1), (2).

3) *$m > m_j$ for all $j=1, 2, \cdots, n$.*

Then $u(t, x)$ vanishes in $[-T, T]\times D$.

Proof. For a positive ε, $0<\varepsilon<1$ fixed, we choose a positive σ so small that $\sum_{j=0}^{m-1} \sum_{j+m|\alpha:\mathfrak{m}|\leq m} A_{j,\alpha}\rho^{|\alpha|}\left(\frac{\sigma T}{\varepsilon}\right)^{m-j}<1-\varepsilon$ holds.

We change the time variable; $t'=\sigma(T-t)$ and we set

$$w(t', x)=u\left(T-\frac{t'}{\sigma}, x\right)$$

and

$$a'_{j,\alpha}(t', x)=(-\sigma)^{m-j}\, a_{j,\alpha}\left(T-\frac{t'}{\sigma}, x\right).$$

We use again the letter t instead of the letter t'. Then we can write (1), (2) in the next;

$$P'w=\left(\frac{\partial^m}{\partial t^m}-\sum_{j=0}^{m-1}\sum_{j+m|\alpha:\mathfrak{m}|\leq m} a'_{j,\alpha}(t, x)\frac{\partial^{j+|\alpha|}}{\partial t^j \partial x^\alpha}\right)w(t, x) \tag{3}$$

$$\frac{\partial^\nu}{\partial t^\nu}w(t_0, x)=0,\ \ t_0=\sigma T,\ \ \nu=0, 1, 2, \cdots, m-1. \tag{4}$$

Setting $v(t, x)=\frac{\partial^m}{\partial t^m}w(t, x)$, we have

$$w(t, x)=\int_{t_0}^{t}\frac{(t_0-\tau)^{m-1}}{(m-1)!}v(\tau, x)d\tau \quad \text{for } 0\leq t\leq t_0.$$

Then (3), (4) are transformed into the following differential-integral equation;

$$v(t, x)-\mathfrak{h}v(t, x)=0, \tag{5}$$

$$\mathfrak{h}=\sum_{j=0}^{m-1}\mathfrak{h}_{m-j},$$

$$\mathfrak{h}_{m-j}=\sum_{m|\alpha:\mathfrak{m}|\leqq m-j} a'_{j,\alpha}(t, x)\frac{\partial^{|\alpha|}}{\partial x^\alpha}\int_{t_0}^{t}\frac{(t-\tau)^{m-j-1}}{(m-j-1)!}v(\tau, x)d\tau.$$

A function g satisfies

(6) $\begin{cases} g \text{ is in } C^{(0,\infty)}_{(t,x)}([-T, T]\times D)^{2)} \text{ and} \\ \text{for any } s,\ \frac{\partial^{|s|}}{\partial x^s}g(t, x) \text{ vanish at } \partial D.^{3)} \end{cases}$

1) This condition means that $a_{j,\alpha}(t, x)$ belongs Gevrey Class of $\left(\frac{m}{m_1}, \frac{m}{m_2}, \cdots, \frac{m}{m_n}\right)$.

2) $C^{(0,\infty)}_{(t,x)}([-T, T]\times D]$ is the set of functions which are continuous in t, and infinitely differentiable in x for (t, x) in $[-T, T]\times D$.

3) ∂D is the boundary of D.

Using such a $g(t, x)$, we can calculate the adjoint operator $g-Hg$ of $v-\mathfrak{h}v$;

$$\int_D\int_0^{t_0} g(v-\mathfrak{h}v)dtdx = gv - \sum_{j=0}^{m-1} \sum_{m|\alpha:\mathfrak{m}|\le m-j} \int_D\int_0^{t_0} a'_{j,\alpha}(t, x)$$
$$\times \frac{\partial^\alpha}{\partial x^\alpha}\left[\int_{t_0}^t \frac{(t_0-\tau)^{m-j-1}}{(m-j-1)!} v(\tau, x)d\tau\right] g(t, x)dtdx$$
$$= gv - \sum_{j=0}^{m-1} \sum_{m|\alpha:\mathfrak{m}|\le m-j} (-1)^\alpha \int_D\int_0^{t_0} \left[\int_{t_0}^t \frac{(t_0-\tau)^{m-j-1}}{(m-j-1)!} v(\tau, x)d\tau\right] \times$$
$$\times \frac{\partial^\alpha}{\partial x^\alpha} \{a'_{j,\alpha}(t, x)\cdot g(t, x)\}\, dtdx.$$

and using the properties

$$a'_{j,\alpha}(t, x)\, g(t, x) = \frac{\partial^{m-j}}{\partial t^{m-j}} \int_0^t \frac{(t-\tau)^{m-j-1}}{(m-j-1)!} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \{a'_{j,\alpha}(\tau, x)\, g(\tau, x)\} d\tau,$$

and

$$\left[\frac{\partial^\mu}{\partial t^\mu} \int_0^t \frac{(t-\tau)^{m-j-1}}{(m-j-1)!} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \{a'_{j,\alpha}(\tau, x)\, g(\tau, x)\}\, d\tau\right]_{t=0} = 0$$

for $0 \le \mu \le m-j-1$,

we continue the above

$$= gv - \sum_{j=0}^{m-1} \sum_{m|\alpha:\mathfrak{m}|\le m-j} (-1)^{|\alpha|+j+m} \int_D\int_0^{t_0} v(t, x) \times$$
$$\times \left[\int_0^t \frac{(t-\tau)^{m-j-1}}{(m-j-1)!} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \{a'_{j,\alpha}(\tau, x)\cdot g(\tau, x)\}\, d\tau\right] dtdx,$$

(7) $$g - Hg = g - \sum_{j=0}^{m-1} H_{m-j} g$$
$$H_{m-j} g = \sum_{m|\alpha:\mathfrak{m}|\le m-j} (-1)^{j+|\alpha|+m} \int_0^t \frac{(t-\tau)^{m-j-1}}{(m-j-1)!} \times$$
$$\times \frac{\partial^{|\alpha|}}{\partial x^\alpha} \{a'_{j,\alpha}(\tau, x)\cdot g(\tau, x)\}\, d\tau.$$

These mean the equality

(8) $$\int_D\int_t^{t_0} g(v-\mathfrak{h}v)dtdx = \int_D\int_t^{t_0} v(g-Hg)\, dtdx.$$

Now we shall show that for any $\psi(t, x)$ satisfying

(9) $$\begin{cases} \psi(t, x) \in C^{(0,\infty)}_{(t,x)}([-t_0, t_0]\times D), \\ \text{for any } s,\ \left|\dfrac{\partial^{|s|}}{\partial x^s}\psi(t, x)\right| \leqq K\rho^{|s|}\Gamma(m\,|\,s:\mathfrak{m}\,|+1) \\ \text{and } \left[\dfrac{\partial^{|s|}}{\partial x^s}\psi(t, x)\right]_{x\in\partial D} = 0 \end{cases}$$

there exists a function $g(t, x)$ satisfying (6) such that

(10) $$g - Hg = \psi$$

holds. The method to solve (10) is due to C. Pucci, De Giorgi and G. Talenti.[4] By a calculate of Gamma function we can obtain

4) See references.

Lemma 1. If $a'_{j,\alpha}(t, x)$ and $f(t, x)$ in $C^{(0;\infty)}_{(t,x)}([-t_0, t_0]\times D)$ satisfy

$$\left|\frac{\partial^{|s|}}{\partial x^s} a'_{j,\alpha}(t, x)\right| \leqq A_{j,\alpha}\sigma^{m-j}\rho^{|s|}\Gamma(m \mid s : \mathfrak{m}\mid +1)$$

$$\left|\frac{\partial^{|s|}}{\partial x^s} f(t, x)\right| \leqq L\rho^{|s|}\Gamma(m \mid s : \mathfrak{m}\mid +l+1)$$

then

$$\left|\frac{\partial^{|s|}}{\partial x^s}\{a'_{j,\alpha}(t, x)\cdot f(t, x)\}\right| \leqq \frac{A_{j,\alpha}L\sigma^{m-j}}{l+1}\rho^{|s|}\Gamma(m \mid s : \mathfrak{m}\mid +l+2)$$

holds.

Lemma 2. For $\psi(t, x)$ satisfying (9)

$$\left|\frac{\partial^{|s|}}{\partial x^s} H_{m-j}\psi\right| \leqq K\Big(\sum_{m\mid\alpha:\mathfrak{m}\mid\leq m-j} A_{j,\alpha}\rho^{|\alpha|}\Big)\frac{t^{m-j}}{(m-j)!}\rho^{|s|}\Gamma(m \mid s : \mathfrak{m}\mid +m-j+2)$$

holds.

This is proved as follow; Taking a term in $H_{m-j}\psi$, we can estimate it by using Lemma 1,

$$\left|\int_0^t \frac{(t-\tau)^{m-j-1}}{(m-j-1)!}\left\{\frac{\partial^{|\alpha|}}{\partial x^\alpha}(A'_{j,\alpha}\psi)\right\}d\tau\right|$$
$$\leq A_{j,\alpha}\sigma^{m-j}\rho^{|\alpha|}K\Gamma(m \mid \alpha : \mathfrak{m}\mid +2)\left|\int_0^t \frac{(t-\tau)^{m-j-1}}{(m-j-1)!}d\tau\right|$$
$$= A_{j,\alpha}\sigma^{m-j}\rho^{|\alpha|}K\Gamma(m \mid \alpha : \mathfrak{m}\mid +2)\frac{t^{m-j}}{(m-j-1)!}\left|\int_0^1 (1-\tilde{s})^{m-j-1}d\tilde{s}\right|.$$

Summing up with respect to α; $m \mid \alpha : \mathfrak{m}\mid \leq m-j$, and replacing $a'_{j,\alpha}\psi$ for $\frac{\partial^{|s|}}{\partial x^s}(a'_{j,\alpha}\psi)$, the lemma is obtained. Repeating these processes of getting Lemma 2 and we can get

Lemma 3. Under the same condition of Lemma 2 and for

$$j_i;\ 0\leq j_i\leq m-1,\ i=1, 2, \cdots, k$$

$$\left|\frac{\partial^{|s|}}{\partial x^s}(H_{m-j_1}H_{m-j_2}\cdots H_{m-j_k}\psi)\right|$$
$$\leqq KA_{j_1}\cdots A_{j_k}\frac{1}{k!}\frac{t^{m-j_1+\cdots+m-j_k}}{(m-j_1+m-j_2+\cdots+m-j_k)!}$$
$$\times\Gamma(m-j_1+\cdots+m-j_k+k+m \mid s : \mathfrak{m}\mid +1)$$

holds, where A_{j_i} denotes $\sum_{m\mid\alpha:\mathfrak{m}\mid\leqq m-j_i} A_{j,\alpha}\rho^{|\alpha|}$.

Using the inequality $\Gamma(u+v+1)\leqq\frac{\Gamma(u+1)\Gamma(v+1)}{\varepsilon^u(1-\varepsilon)^v}$ for any $u\geqq 0$, $v\geqq 0$ and any fixed ε; $0<\varepsilon<1$, we can further estimate

$$\left|\frac{\partial^{|s|}}{\partial x^s}H^k\psi\right| = \left|\frac{\partial^{|s|}}{\partial x^s}\Big(\sum_{j=0}^{m-1}H_{m-j}\psi\Big)^k\right| \leqq K\Big(\sum_{j=0}^{m-1}\sum_{m\mid\alpha:\mathfrak{m}\mid\leq m-j}A_{j,\alpha}\rho^{|\alpha|}\Big(\frac{t}{\varepsilon}\Big)^{m-j}$$
$$\times\frac{1}{1-\varepsilon}\Big)^k\left[\frac{\rho}{(1-\varepsilon)^{m\mid 1:\mathfrak{m}\mid}}\right]^{|s|}\Gamma(m \mid s : \mathfrak{m}\mid +k+1).$$

For $a\geqq 0$, $|z|<1$, the power series $\sum_{k=0}^{+\infty}\frac{\Gamma(a+k+1)}{k!}z^k$ converges. Therefore $\sum_{k=0}^{+\infty}\frac{\partial^{|s|}}{\partial x^s}H^k\psi$ converges absolutely uniformly on $[-t_0, t_0]\times D$, if ε is choson $A_{j,\alpha}\frac{\rho^{|\alpha|}}{1-\varepsilon}\left(\frac{t_0}{\varepsilon}\right)^{m-j}<1$. Thus

(11) $$g=\sum_{k=0}^{+\infty}H^k\psi \text{ is the required one.}$$

As the left hand side of (8) is vanish, if we proved that (10) is solved for any continuous function ψ on $[-t_0, t_0]\times D$ satisfying $[\psi(t, x)]_{x\in\partial D}=0$, v is vanish on $[-t_0, t_0]\times D$. By the condition (2) and (4), the solution u of (1), (2) vanishes there. Therefore to finish the proof of Theorem, only the next Lemma must be proved.

Lemma 4. The linear hull of the functions satisfying (9) is dense in the set of continuous functions on $[-t_0, t_0]\times D$ with respect to the topology of uniform convergence.

Proof of Lemma. Let be $D_\varepsilon=\{x\in D;\ \text{distance }(x, D^c)^{5)}\geqq\varepsilon\}$. φ_ε is defined in the following,

$$\varphi_\varepsilon(x)=\begin{cases}1 & x\in D_{2\varepsilon}\\ 0 & x\in D_\varepsilon^c\end{cases}\ \text{and } 0\leq\varphi_\varepsilon\leqq 1 \text{ and in } C_x^\infty(D).$$

For $\psi(t, x)$ satisfying (9), we set $\psi_\varepsilon(t, x)=\varphi_\varepsilon(x)\cdot\psi(t, x)$. Then $\psi_\varepsilon(t, x)$ converges uniformly to $\psi(t, x)$ on $[-t_0, t_0]\times D$. $J_a(x)$ is defined as follow,

$$J_a(x)=\begin{cases}e^{-2x^{-a}} & x>0\\ 0 & x\leq 0\end{cases}$$

which is in $C_x^\infty(R^1)$ and satisfies

$$\left|\frac{\partial^s}{\partial x^s}J_a(x)\right|\leq M\tilde{\rho}^s\Gamma\left(\left(1+\frac{1}{a}\right)s+1\right) \text{ for some } M,\ \tilde{\rho}.$$

Using this, $J_{a,\eta}(x)$ is defined as follows:

$J_{a,\eta}(x)=C\prod_{j=1}^{n}J_a(\eta+x_j)\,J_a(\eta-x_j)$ and the constant C is determined as $\int_{R^n}J_{a,\eta}(x)dx=1$. Then the convolution $(\psi_\varepsilon^* J_{a,\eta})(x)$ satisfies

$$\left|\frac{\partial^{|s|}}{\partial x^s}(\psi_\varepsilon * J_{a,\eta})(x)\right|\leqq M'\tilde{\rho}^{|s|}\Gamma\left(|s|\left(1+\frac{1}{a}\right)+1\right)\quad\text{for } \varepsilon>\eta>0.$$

For one b; $1<b<\min_{1\leq j\leq n}\frac{m}{m_j}$, if we choose the above a to satisfy $b>1+\frac{1}{a}$, $(\psi_\varepsilon * J_{a,\eta})(x)$ satisfies (9) and converges to $\psi(t, x)$ uniformly on $[-t_0, t_0]\times D$ as ε and η tending to zero keeping $\varepsilon>\eta$. This proves Lemma. These processes prove Theorem.

---

5) D^c is the complement of D.

References

[1] H. Kumanogo: On the uniqueness for the solution of the Cauchy problem. Osaka Math. Jour., **15**, 151-172 (1963).

[2] A. Tsutsumi: On the uniqueness of the Cauchy problem for semi-elliptic part. diff. equations. I, II and III. Proc. Japan Acad., **39**, 781-785, 786-790 (1963), and loc. cit., **40**, 259-261 (1964).

[3] De Giorgi: Un teorema di unicita per il problema di Cauchy relativo ad equa. diff. lin. a deri. part. tipo parabolico. Annali di mate. pura ed appli. ser. iv, **40**, 371-377 (1955).

[4] G. Talenti: Un problema di Cauchy. Annali della Scoul. Norm. Sup. Pisa., 165-186 (1964).

[5] M. Pucci: Nuove ricerche sul problema di Cauchy. Mem. Acad. Scie. Torino., 45-67 (1953).

[6] Y. Ohya: Existence des solutions de classes superieure dans les equations differentielles faiblement hyperboliques à coefficients variables. Proc. Japan Acad., **38**, 726-730 (1962).