3. *Extensions of Topologies*

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Let (X, τ) be a topological space and $\tau \subset \tau^*$. Then τ^* will be called a simple extension of τ if and only if there exsists an $A \subset X$ such that $\tau^* = \{O \cup (O' \cap A) \mid 0, 0' \in \tau\}$. In this case we write $\tau^* = \tau(A)$. This definition is due to N. Levine [2]. N. Levine has obtained some interesting results about simple extensions of topologies [2].

It is the purpose of this note to consider the simple extensions of regular or other several topologies. In the next, we shall consider a generalization of simple extensions in § 3.

Let (X, τ) be a topological space and $\tau^* = \tau(A)$. Then we shall notice that for each $x \notin A$, the τ-open neighborhood system of x is a τ^*-open base of x and for each $x \in A$, the family $\{V(x) \cap A \mid V(x)$: τ-open neighborhood of $x\}$ is a τ^*-open base of x. Thus it is sufficient to consider these open bases.

The notations which will be used in this note are chiefly following. A^c denotes the complement of A. $\bar{A}$ and $\bar{A}^*$ denote the closure operators relative to τ and τ^* respectively. By $U(x)$, $V(x)$, and $W(x)$ we denote τ-open neighborhoods of x. $(A, \tau \cap A)$ denotes the subspace A of (X, τ), that is, $\tau \cap A$ denotes the relative topology of A with respect to τ.

The following facts have been shown in Lemma 3 of [2]. Let (X, τ) be a topological space and $\tau^* = \tau(A)$. Then $(A, \tau \cap A) = (A, \tau^* \cap A)$ and $(A^c, \tau \cap A^c) = (A^c, \tau^* \cap A^c)$. This follows from the above remark about the τ^*-open base of x.

§ 1. Simple extensions of regular topologies. In this section, we shall obtain a result about simple extensions of regular topologies which is better than N. Levine's theorem [2] and its application.

Let (X, τ) be a topological space and A a subset of X. We shall say that A is R-open in (X, τ) if and only if for each $x \in A$, there exists a $V(x)$ such that $V(x) \cap \bar{A} \subset A$, i.e., A is open in $(\bar{A}, \tau \cap \bar{A})$.

Theorem 1.1. *Let (X, τ) be a regular space and $\tau^* = \tau(A)$. Then the following conditions* (i)~(iii) *are equivalent*:

(i) *A is R-open in (X, τ);*

(ii) *$\bar{A} \cap A^c$ is closed in (X, τ);*

(iii) *(X, τ^*) is regular.*

Proof. It is evident that (i) and (ii) are equivalent. Then we shall only prove that (i) and (iii) are equivalent.

(i)$\to$(iii): Let $x\notin A$ and let $V(x)$ be an arbitrary τ^*-neighborhood of x. Since $V(x)$ is also a τ-neighborhood of x and (X, τ) is regular, there exists a $U(x)$ such that $\overline{U(x)}\subset V(x)$. $U(x)$ is a τ^*-neighborhood of x and since $\tau\subset\tau^*$, $\overline{U(x)}^*\subset\overline{U(x)}\subset V(x)$. Let $x\in A$ and $V(x)\cap A$ an arbitrary τ^*-neighborhood of x. From (i), there exists a $U(x)$ such that $U(x)\subset V(x)$ and $U(x)\cap\bar{A}\subset A$. By regularity of (X, τ), there exists $W(x)$ such that $\overline{W(x)}\subset U(x)$. Then $W(x)\cap A$ is a τ^*-neighborhood of x and $\overline{W(x)\cap A}^*\subset \overline{W(x)}\cap A\subset U(x)\cap\bar{A}=U(x)\cap A\subset V(x)\cap A$. Hence the regularity of (X, τ^*) is proved.

(iii)$\to$(i): Assumed that A is not R-open in (X, τ). Then there exists a point $x\in A$ such that for any $V(x)$, $V(x)\cap\bar{A}\not\subset A$. We shall next prove that for each τ^*-neighborhood $V(x)\cap A$, $\overline{V(x)\cap A}^*\not\subset A$. Since $V(x)\cap\bar{A}\not\subset A$, there exists a point $y\in V(x)\cap\bar{A}-A$. Hence $y\notin A$ and τ-neighborhood system of y is a τ^*-open base of y. Since $y\in V(x)$ and $y\in\bar{A}$, for any $U(y)$, $U(y)\cap V(x)\cap A\neq\phi$. Hence $y\in\overline{V(x)\cap A}^*$. Thus (X, τ^*) is not regular. This completes the proof.

This theorem is a generalization of Theorems 2 and 3 in [2].

Corollary 1.1. *Let (X, τ) be a regular space. If $(X, \tau(A))$ and $(X, \tau(B))$ are regular, then $(X, \tau(A\cap B))$ is regular.*

Proof. From the condition (ii) of Theorem 1.1, we can easily see that $A\cap B$ is also R-open in (X, τ).

Under the same conditions of Corollary 1.1, we can easily see that $(X, \tau(A\cup B))$ and $(X, \tau(A^c))$ are not necessarily regular.

Theorem 1.2. *Let (X, τ) be a completely regular space and $\tau^*=\tau(A)$. Then a necessary and sufficient condition that (X, τ^*) be completely regular is that A is R-open in (X, τ).*

Proof. The necessity is obvious from Theorem 1.1. Let A be R-open in (X, τ). Case 1: $x\notin A$. Let $V(x)$ be an arbitrary τ^*-neighborhood of x. Since $V(x)$ is, of course, a τ-neigborhood of x, there exists a continuous mapping f from (X, τ) into the closed interval $[0, 1]$ such that $f(x)=0$ and $f(V(x)^c)=1$. Since $\tau\subset\tau^*$, f is continuous in (X, τ^*). Case 2: $x\in A$. Let $V(x)\cap A$ be an arbitrary τ^*-neighborhood of x. Since (X, τ^*) is regular by Theorem 1.1, there exists an $U(x)\cap A$ such that $\overline{U(x)\cap A}^*\subset V(x)\cap A$. Since $(A, \tau\cap A)=(A, \tau^*\cap A)$, $(A, \tau^*\cap A)$ is completely regular (because any subspace of a completely regular space is completely regular). Hence there is a continuous mapping h from $(A, \tau^*\cap A)$ into $[0, 1]$ such that $h(x)=0$ and $h(A-U(x)\cap A)=1$. We define a mapping f so that for $y\in A$, $f(y)=h(y)$ and for $y\notin A$, $f(y)=1$. Since $\overline{U(x)\cap A}^*\subset A$, it follows that f is a continuous mapping from (X, τ^*) into $[0, 1]$. Clearly $f(x)=0$ and $f(X-V(x)\cap A)=1$. Therefore (X, τ^*) is completely regular. This completes the proof.

This theorem is a generalization of Theorem 4 in [2].

Theorem 1.3. *Let (X, τ) be a metrizable space and $\tau^*=\tau(A)$. In order that (X, τ^*) be metrizable, it is necessary and sufficient that A be R-open in (X, τ).*

Proof. The necessity is obvious from Theorem 1.1. Let A be R-open in (X, τ). Since (X, τ) is T_1 and regular, (X, τ^*) is also T_1 and regular. Nagata-Smirnov theorem asserts that regular T_1-space is metrizable if and only if its topology has a σ-locally finite open base (cf. [3], [4]). The topology τ has a σ-locally finite open base $\mathfrak{B}=\bigcup_{i=1}^{\infty}\mathfrak{B}_i$, where each $\mathfrak{B}_i$ is a locally finite collection of open sets. It is evident that $\mathfrak{B}\smile\bigcup_{i=1}^{\infty}\{V\cap A\,|\,V\in\mathfrak{B}_i\}$ is an open base of (X, τ^*) and $\mathfrak{B}_i$ and $\{V\cap A\,|\,V\in\mathfrak{B}_i\}$ are locally finite collections of τ^*-open sets. Therefore, (X, τ^*) is metrizable. This completes the proof.

§ 2. Simple extensions of several topologies.

Theorem 2.1. *Let (X, τ) be a connected space and $\tau^*=\tau(A)$. If A is dense in (X, τ), then (X, τ^*) is connected.*

Proof. Assume that (X, τ^*) is not connected. Then there exist two non-void τ^*-open sets G_1 and G_2 such that $G_1\cup G_2=X$ and $G_1\cap G_2=\phi$. On the other hand, (X, τ) is connected and hence either G_1 or G_2 is not open in (X, τ), say G_1. Then there is a point $x\in G_1$ such that for any $V(x)$, $V(x)\cap G_2\neq\phi$. Case 1: $x\notin A$. Since $V(x)$ is any τ^*-neighborhood of x, $x\in\bar{G}_2^*$ which contradicts. Case 2: $x\in A$. Now there is a point $y\in V(x)\cap G_2$ for each $V(x)$. If $y\notin A$, then there is a τ^*-neighborhood $V(y)$ such that $V(y)\subset G_2$, since G_2 is τ^*-open. Since $V(x)\cap V(y)$ is a τ-neighborhood of y and $\bar{A}=X$, $\phi\neq(V(x)\cap V(y))\cap A\subset(V(x)\cap A)\cap G_2$. If $y\in A$, then there is a $V(y)\cap A$ such that $V(y)\cap A\subset G_2$. In the same way, $\phi\neq(V(x)\cap V(y))\cap A\subset(V(x)\cap A)\cap G_2$. Hence $x\in\bar{G}_2^*$ which is a contradiction. This completes the proof.

Since the conditions of N. Levine's theorem [2, Theorem 9] is that A is dense and connected in (X, τ) (hence (X, τ) is connected), this theorem is a generalization of Theorem 9 in [2].

Let (X, τ) be a topological space and $\tau^*=\tau(A)$. Now we shall consider the case $A^c\in\tau$. Then A is clearly open and closed in (X, τ^*). Hence (X, τ^*) is the union of two disjoint open and closed subspaces $(A, \tau\cap A)$ and $(A^c, \tau\cap A^c)$. In general, let P be a topological property satisfying the following conditions:

(1) If (X, τ) has property P, then any open (or closed) subspace of (X, τ) has property P;

(2) Let A and B be separated sets in (X, τ), i.e., $\bar{A}\cap B=A\cap\bar{B}=\phi$. If the subspaces $(A, \tau\cap A)$ and $(B, \tau\cap B)$ have property

P, then $(A\cup B, \tau\cap(A\cup B))$ has property P.

Theorem 2.2. *Let (X, τ) be a topological space and $\tau^*=\tau(A)$ and $A^c\in\tau$. The space (X, τ^*) has property P if and only if $(A, \tau\cap A)$ and $(A^c, \tau\cap A^c)$ have property P.*

Proof. Let (X, τ^*) has property P. Since A and A^c is open and closed in (X, τ^*), $(A, \tau\cap A)$, and $(A^c, \tau\cap A^c)$ have property P from (1). If $(A, \tau\cap A)$ and $(A^c, \tau\cap A^c)$ have property P. Since A and A^c are separated in (X, τ^*), $(A\cup A^c, \tau^*\cap(A\cup A^c))=(X, \tau^*)$ has property P from (2).

Almost all topological properties except the connectness satisfy property P. Therefore, Theorem 2.2 is a generalization of Theorems 2, 4 and 5 in [2].

§ 3. **Extensions of regular or connected topologies.** In this section, we shall consider a generalization of simple extensions, i.e., ordinary extensions of topologies.

Let (X, τ) a topological space and let $\mathfrak{A}=\{A_\alpha\}$ be a collection of subsets of X. We define a topology $\tau\mathfrak{A}$ which shall be called a extension of τ by $\mathfrak{A}$ in the following way. For $x\notin\bigcup_\alpha A_\alpha$, a new open base of x is the original open neighborhood system of x and for $x\in\bigcup_\alpha A_\alpha$, a new open base of x is the family which consists of all intersections of an original open neighborhood of x and an intersection of any finite number of sets (containing x) of $\mathfrak{A}$.

Let (X, τ) be a topological space and let $\mathfrak{A}$ be a collection of subsets of X. For convenience, by F, G and H we shall represent the intersections of any finite number of sets of $\mathfrak{A}$. We shall say that $\mathfrak{A}$ is R-open in (X, τ) if and only if for each F and for each $x\in F$, there exists a $V(x)$ and a G containing x so that for each $y\in V(x)\cap\bar{G}-F$, there exists an H containing y such that $y\notin\overline{H\cap G}$.

Since the intersection of a finite number of R-open sets is also R-open from Corollary 1.1, it is evident that the collection which consists of R-open sets is R-open. But the converse is false. Because, for any subset $A\subset X$, the collection $\{A, A^c\}$ is R-open. If for any α, the collection $\mathfrak{A}_\alpha$ is R-open, then a collection $\bigcup_\alpha\mathfrak{A}_\alpha$ is R-open.

Theorem 3.1. *Let (X, τ) be a regular space and let $\mathfrak{A}=\{A_\alpha\}$ be a collection of subsets of X. The space $(X, \tau\mathfrak{A})$ is regular if and only if $\mathfrak{A}$ is R-open in (X, τ).*

Proof. Sufficiency. Suppose that $\mathfrak{A}$ is R-open in (X, τ). For $x\notin\bigcup_\alpha A_\alpha$, the regularity at x is obvious. For $x\in\bigcup_\alpha A_\alpha$, let $V(x)\cap F$ be an arbitrary $\tau\mathfrak{A}$-neighborhood of x. From the assumption, there exists a $U(x)\cap G$ such that $U(x)\cap G\subset V(x)\cap F$ and it satisfies the condition of the above definition. Since (X, τ) is regular, there is a

$W(x)$ such that $\overline{W(x)} \subset U(x)$. Then $W(x) \cap G$ is a $\tau\mathfrak{A}$-neighborhood of x. We shall show that $\overline{W(x) \cap G}^{\circ} \subset V(x) \cap F$, where $\bar{M}^{\circ}$ denotes the closure of M in $(X, \tau\mathfrak{A})$. Since $\overline{W(x) \cap G}^{\circ} \subset \overline{W(x) \cap G} \subset U(x) \cap \bar{G}$ and for any point $y \in \overline{W(x) \cap G}^{\circ}$ and for any $V(y) \cap H (y \in H)$, $\phi \neq (V(y) \cap H) \cap (W(x) \cap G) \subset V(y) \cap (H \cap G)$, it follows that $y \notin V(x) \cap \bar{G} - F$, i.e., $y \in F$. Hence $\overline{W(x) \cap G}^{\circ} \subset V(x) \cap F$.

Necessity. Assume that $\mathfrak{A}$ is not R-open in (X, τ). Then there exist F and $x \in F$ which satisfy the following conditions. Let $V(x) \cap G$ be an arbitrary $\tau\mathfrak{A}$-neighborhood of x. Then there exists a point $y \in V(x) \cap \bar{G} - F$ such that for any $H(\ni y)$, $y \in \overline{H \cap G}$, i.e., for any neighborhood $V(y) \cap H$, $(V(y) \cap H) \cap (V(x) \cap G) \neq \phi$. Hence $y \in \overline{V(x) \cap G}^{\circ}$. Since $y \notin F$, $\overline{V(x) \cap G}^{\circ} \not\subset F$. Therefore $(X, \tau\mathfrak{A})$ is not regular. This completes the proof.

This theorem is a generalization of Theorem 1.1, since if a collection which consists of only one set is R-open, then its set is R-open. The definition of R-open collection is complicated and not beautiful. However, this complicated condition is required even in the case when the collection consists of only two sets.

Theorem 3.2. *Let (X, τ) be a metrizable space and let $\mathfrak{A}$ be a σ-locally finite collection of subsets of X. The space $(X, \tau\mathfrak{A})$ is metrizable if and only if $\mathfrak{A}$ is R-open in (X, τ).*

Proof. In the same way as Theorem 1.3, it is sufficient to show that $\tau\mathfrak{A}$ has a σ-locally finite open base, but it is easily seen from the fact that the family consisting of all intersection of any finite number of sets of $\mathfrak{A}$ is σ-locally finite.

This theorem is a generalization of Theorem 1.3, but if we omit the "σ-locally finite", then this assertion is false. For example, let $X = \{(x, y) \mid y \geqq 0\}$ and let τ be the usual topology on X. Let $A_{pn} = \{(x, y) \mid (x-p)^2 + (y-n^{-1})^2 < n^{-2}\} \cup \{(p, 0)\}$ and $\mathfrak{A} = \{A_{pn} \mid p :$ real number, $n = 1, 2, \cdots\}$. Then $\mathfrak{A}$ is R-open and not σ-locally finite. The space $(X, \tau\mathfrak{A})$ is well known as the example which is regular and not normal (hence not metrizable) (cf. [1, p. 133, I]).

Theorem 3.3. *Let (X, τ) be a connected space and let $\mathfrak{A}$ be a collection of subsets of X. If each intersection of any finite number of sets of $\mathfrak{A}$ is dense in (X, τ), then $(X, \tau\mathfrak{A})$ is connected.*

Proof. Assume that $(X, \tau\mathfrak{A})$ is not connected. Then there exist two non-void $\tau\mathfrak{A}$-open set O_1 and O_2 such that $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \phi$. But (X, τ) is connected and hence either O_1 or O_2 is not τ-open, say O_1. Then there exists $x \in O_1$ such that for any $V(x)$, $V(x) \cap O_2 \neq \phi$. Let $V(x) \cap F$ be an arbitrary neighborhood of x. For every $V(x)$, there exists $y \in V(x) \cap O_2$. Since $y \in O_2$ which is $\tau\mathfrak{A}$-open, for some $V(y) \cap G$ $(y \in G)$, $V(y) \cap G \subset O_2$. Since $F \cap G$ is dense in (X, τ), $\phi \neq$

$(V(x)\cap V(y))\cap(F\cap G)\subset(V(x)\cap F)\cap O_2$. Hence $x\in \bar{G}_2^\circ$ which is a contradiction. This completes the proof.

This theorem is a generalization of Theorem 2.1.

References

[1] J. L. Kelley: General Topology. Van Nostrand, New York (1955).
[2] N. Levine: Simple extensions of topologies. Amer. Math. Manth., **71**, 22-25 (1964).
[3] J. Nagata; On a necessary and sufficient condition of metrizability. J. Inst. Polytech. Osaka City Univ., **1**, 93-100 (1950).
[4] Yu. M. Smirnov: A necessary and sufficient condition of a topological space. Doklady Akad. Nauk S.S.S.R. N.S., **77**, 197-200 (1951).