# 1. *Positive Pseudo-resolvents and Potentials*

By Kôsaku YOSIDA
Department of Mathematics, University of Tokyo
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1. **Introduction.** Let $\Omega$ be a set, and denote by $X$ a Banach space of real-valued bounded functions $f(x)$ defined on $\Omega$ and normed by $\|f\|=\sup_{x\in\Omega}|f(x)|$. We assume that $X$ is closed with respect to the lattice operations $(f\wedge g)(x)=\min(f(x), g(x))$ and $(f\vee g)(x)=\max(f(x), g(x))$. For any linear subspace $Y$ of $X$, we shall denote by $Y^+$ the totality of functions $f\in Y$ which are $\geqq 0$ on $\Omega$, in symbol $f\geqq 0$. We also use the notation $f^+=f\vee 0$ and $f^-=(-f)\vee 0$.

We denote by $L(X, X)$ the totality of continuous linear operators defined on $X$ into $X$. A family $\{J_\lambda; \lambda>0\}$ of operators $\in L(X, X)$ is called a *pseudo-resolvent* if it satisfies the *resolvent equation*

(1) $$J_\lambda - J_\mu = (\mu-\lambda)J_\lambda J_\mu.$$

Suggested by the case of the resolvent $J_\lambda=(\lambda I-A)^{-1}$ of the infinitesimal generator $A$ of a semi-group $\{T_t; t\geqq 0\}$ of operators $\in L(X, X)$ of class $(C_0)$[1)] mapping $X^+$ into $X^+$, we shall assume conditions:

(2) $J_\lambda$ is *positive*, in symbol $J_\lambda\geqq 0$, that is, $f\geqq 0$ implies $J_\lambda f\geqq 0$ for all $\lambda>0$.

(3) $$\|\lambda J_\lambda\|\leqq 1 \quad \text{for all } \lambda>0.$$

Then, an element $f\in X$ is called *superharmonic* (or *subharmonic*) if $\lambda J_\lambda f\leqq f$ (or $\lambda J_\lambda f\geqq f$) for all $\lambda>0$, and an element $f\in X$ is called a *potential* if there exists a $g\in X$ such that $f=s\text{-}\lim_{\lambda\downarrow 0} J_\lambda g$, where $s$-lim denotes the strong limit in $X$, i.e., uniform limit on $\Omega$.

We shall be concerned with the *potential operator* $V$ defined by

(4) $Vf=s\text{-}\lim_{\lambda\downarrow 0} J_\lambda f$ (when $s\text{-}\lim_{\lambda\downarrow 0} J_\lambda f^+$ and $s\text{-}\lim_{\lambda\downarrow 0} J_\lambda f^-$ both exist).

Our main results are stated in the following two theorems.

**Theorem 1.** Let $J_\lambda$ satisfy (1) and (2). Then $V\geqq 0$ and we have:

(5) Let $f\in X^+, g\in X^+$ and $\lambda>0$, and define $V_\lambda=V+\lambda^{-1}I$. If $(V_\lambda f)(x)\leqq(Vg)(x)$ on the support $(f)$, we must have $V_\lambda f\leqq Vg$. (*the principle of majoration*).

**Theorem 2.** Let $J_\lambda$ satisfy (1), (2) and (3). If the *range* $R(V)$ of the potential operator $V$ is dense in $X$, then $R(V_\lambda)$ is also dense in $X$ and the *null space* $N(V)=\{f; Vf=0\}$ consists of the zero vector only. Moreover, $J_\lambda$ is the resolvent of a linear operator $A$ with dense *domain* $D(A)$ defined through the *Poisson equation* $AVf=-f$.

**Remark.** Two special cases of $X$ are important for concrete

1) See, e.g., K. Yosida: Functional Analysis, Springer, to appear soon.

application. *The first case*: $\Omega$ is a locally compact Hausdorff space and $X$ is the totality of real-valued continuous functions defined on $\Omega$ which *tend to zero at infinity*; $X$ is the closure with respect to the norm $\|f\| = \sup_{x\in\Omega} |f(x)|$ of the space $C_0(\Omega)$ of continuous functions with compact support defined on $\Omega$. *The second case*: $\Omega$ is a $\sigma$-additive family of subsets of a set, and $X$ is the Banach space of $\sigma$-additive measures defined on $\Omega$ and normed by the total variation of the measure. The first case was discussed by G. A. Hunt[2] in the view to characterize the operator $\tilde{V}$ defined through

$$(4)' \qquad (\tilde{V}f)(x) = \int_0^\infty (T_t f)(x)dt,$$

where $\{T_t;\, t \geqq 0\}$ is the semi-group associated with a Markov process in a locally compact space $\Omega$. *In the first case*, we can prove, under condition(5), an analogue of Hunt's research:

**Theorem 3.** If $V \geqq 0$ and if $R(V)$ is dense in $X$, we have:

$(5)_1$ Let $f \in C_0(\Omega)^+$ and $g \in X^+$, and let $(Vf)(x) \leqq (Vg)(x)$ on the support $(f)$. Then $Vf \leqq Vg$.

If furthermore, $V(C_0(\Omega)^+)$ is dense in $X^+$, then we obtain:

$(5)_2$ Let $f \in C_0(\Omega)$ and let $(Vf)(x_0) = \max_{x\in\Omega} (Vf)(x)$. Then $f(x_0) \geqq 0$.

**Theorem 4.** Let $V$ be a closed linear operator whose domain $D(V)$ and range $R(V)$ both belong to $X = C_0(\Omega)^a$ in such a way that $V$ satisfies (5) and further conditions:

$$(6) \qquad V \geqq 0,$$

(7) $Vf$ is defined if and only if $Vf^+$ and $Vf^-$ are both defined.

$$(8) \qquad N(V) = \{0\}.$$

$$(9) \qquad C_0(\Omega) \subseteqq D(V).$$

(10) $V(C_0(\Omega)^+)$ is dense in $X^+$ and $V_\lambda(C_0(\Omega))$ is dense in $X$ for $\lambda > 0$.

Then, for the operator $A$ defined through the Poisson equation $AVf = -f$ and for $\lambda > 0$, the resolvent $J_\lambda = (\lambda I - A)^{-1}$ exists as an operator $\in L(X, X)$ such that (1), (2), (3) and (4) hold.

**2. Proof of the theorems.** We shall rely upon a lemma which is a special case of the so-called *Abelian ergodic theorem*.[3]

**Lemma.** Under condition (1), we have

$$(11) \qquad J_\lambda J_\mu = J_\mu J_\lambda.$$

Under conditions (1) and (3), we have:

(12) $R(J_\lambda)$ is independent of $\lambda$, and its closure $R(J_\lambda)^a$ coincides with $\{f;\, s\text{-}\lim_{\lambda\to\infty} \lambda J_\lambda f = f\}$.

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2) Markoff processes and potentials, I, II and III, Illinois J. of Math., **1**, 44–93, 316–469 (1957) and **2**, 151–213 (1958). Further researches are given, e.g., in Séminaire du Potentiels, dirigés par M. Brelot, G. Choquet et J. Deny, Fac. Sci. Paris (1950–).

3) K. Yosida: Ergodic theorems for pseudo-resolvents. Proc. Japan Acad., **37**, 422–423 (1961). Cf. E. Hille-R. S. Phillips, Functional Analysis and Semi-groups, Providence (1957).

(12)′ $R(I-\lambda J_\lambda)$ is independent of $\lambda$ and its closure $R(I-\lambda J_\lambda)^a$ coincides with $\{f;\ s\text{-}\lim_{\lambda\downarrow 0} \lambda J_\lambda f=0\}$.

**Proof.** See the reference cited in the footnotes 1) and 3).

**Proof of Theorem 1.** The operator $V$ defined through (4) satisfies

(13) $$Vf=\lambda J_\lambda Vf+J_\lambda f=V\lambda J_\lambda f+J_\lambda f \quad \text{for } f\in D(V),$$

because of (1), (4) and (11). Thus, if $f\geqq 0$ belongs to the domain $D(V)$, then $Vf$ is superharmonic by $J_\lambda f\geqq 0$.

Next we show that, if $f\in X^+$ be such that $\mu J_\mu f\leqq f$ for all $\mu$ with $0<\mu\leqq\lambda$, then

(14) $$\lim_{\mu\to 0}(J_\mu(\lambda I-\lambda^2 J_\lambda)f)(x)=(\lambda J_\lambda f)(x)-f_h(x), \text{ where}$$
$$f_h(x)=\lim_{\mu\downarrow 0}(\mu J_\mu f)(x)\text{[4]}.$$

To prove this, we first observe that $\lambda(I-\lambda J_\lambda)f$ is $\geqq 0$. We have, by (1),

$$J_\mu(\lambda f-\lambda^2 J_\lambda f)=\lambda J_\mu f-\frac{\lambda^2}{\lambda-\mu}(J_\mu-J_\lambda)f=\frac{-\lambda}{\lambda-\mu}\mu J_\mu f+\frac{\lambda^2}{\lambda-\mu}J_\lambda f.$$

We also have, by (1),

$$(I+(\mu-\lambda)J_\lambda)(I-\mu J_\mu)f=(I-\lambda J_\lambda)f.$$

Hence, if $0<\mu\leqq\lambda$, the condition $\mu J_\mu f\leqq f$ (for $0<\mu\leqq\lambda$) implies that $0\leqq\mu J_\mu f\leqq\lambda J_\lambda f$. Thus $\lim_{\mu\downarrow 0}(\mu J_\mu f)(x)=f_h(x)$ exists and so we obtain (14).

We are now able to prove (5). Put $v(x)=\min((V_\lambda f)(x), (Vg)(x))$. Then $v\geqq 0$ by $f\geqq 0$, $g\geqq 0$ and $V\geqq 0$, and we have

(15) $$\mu J_\mu v\leqq v \qquad \text{for} \quad 0<\mu\leqq\lambda.$$

We have only to show that $\mu J_\mu V_\lambda f\leqq V_\lambda f$. But we obtain

$$\mu J_\mu V_\lambda f=\mu J_\mu Vf+\frac{\mu}{\lambda}J_\mu f=\mu J_\mu Vf+J_\mu f+\left(\frac{\mu}{\lambda}-1\right)J_\mu f$$
$$=Vf+\left(\frac{\mu}{\lambda}-1\right)J_\mu f\leqq Vf\leqq V_\lambda f.$$

Thus $w=\lambda(I-\lambda J_\lambda)v\geqq 0$ and we have, by (14),

(16) $$\lim_{\mu\downarrow 0}(J_\mu w)(x)=(\lambda J_\lambda v)(x)-v_h(x), \text{ where } v_h(x)=\lim_{\mu\downarrow 0}(\mu J_\mu v)(x)\geqq 0.$$

Hence, by (13) and the positivity of $J_\mu$, we obtain

(17) $$\lim_{\mu\downarrow 0}(J_\mu w)(x)\leqq(\lambda J_\lambda v)(x)=v(x)-\lambda^{-1}w(x)$$
$$\leqq(\lambda J_\lambda V_\lambda f)(x)=(Vf)(x)$$
$$=(V_\lambda f)(x)-\lambda^{-1}f(x).$$

We have $(V_\lambda f)(x)=v(x)$ on the support $(f)$ by hypothesis. Hence, by (17), $f(x)\leqq w(x)$ on the support $(f)$, and so, by $f\geqq 0$ and $w\geqq 0$, we must have $f\leqq w$. Therefore, by (17) and the positivity of $J_\mu$, we obtain $(V_\lambda f)(x)\leqq \lim_{\mu\downarrow 0}(J_\mu w)(x)+\lambda^{-1}w(x)\leqq v(x)\leqq (Vg)(x)$, that is, $V_\lambda f\leqq Vg$.

**Proof of Theorem 2.** By (13), we see that $R(V)^a=X$ implies $R(V_\lambda)^a=X$ and $R(J_\lambda)^a=X$. $R(J_\lambda)^a=X$ implies, by (12), that $N(J_\lambda)=$

4) Originally, the author tacitly concluded that $s\text{-}\lim_{\lambda\downarrow 0}\lambda J_\lambda f=f_h$ exists. This was pointed out by Mr. D. Fujiwara.

$\{0\}$ which, in turn, implies the existence of the inverse $J_\lambda^{-1}$. By (1), it is easy to see that $(\lambda I - J_\lambda^{-1})$ is independent of $\lambda$ so that $J_\lambda = (\lambda I - A)^{-1}$ where $A = \lambda I - J_\lambda^{-1}$. Moreover, $D(A) = R(J_\lambda) \supseteqq R(V)$ is dense in $X$. By (13), $Vf = 0$ implies $J_\lambda f = 0$ so that $N(V) = \{0\}$ if $R(V)^a = X$. Finally, we have, by (13) and $J_\lambda = (\lambda I - A)^{-1}$,

$$(\lambda I - A) Vf = \lambda Vf + f, \text{ that is, } AVf = -f.$$

**Proof of Theorem 3.** We first prove $(5)_1$. Since $R(V)$ is dense in $X$, there exists an $h \geqq 0$ such that $f(x) \leqq (Vh)(x)$ on the support $(f)$ which is compact by hypothesis. For any $\varepsilon > 0$, take $\lambda > 0$ such that $\lambda^{-1} < \varepsilon$. Then $(V_\lambda f)(x) \leqq (Vg)(x) + \lambda^{-1} f(x) \leqq (V(g + \varepsilon h))(x)$ on the support $(f)$. Hence, by (5), $Vf \leqq V_\lambda f \leqq V(g + \varepsilon h)$. Letting $\varepsilon \downarrow 0$, we obtain $Vf \leqq Vg$.

Proof of $(5)_2$. Since $Vf \in X = C_0(\Omega)^a$, we must have $(Vf)(x_0) \geqq 0$. Let us tentatively assume that $(Vf)(x_0) > 0$. Then we can show that $f(y) \geqq 0$ at some point $y \in E = \{x; (Vf)(x_0) = (Vf)(x)\}$. Since $V \geqq 0$, the condition $(Vf)(x_0) > 0$ implies that $f^+$ is not equal to zero. For any point $y \in E \cap \text{support}(f^+)$, we have surely $f(y) \geqq 0$. If $E \cap \text{support}(f^+)$ is void, then there exists an $\varepsilon > 0$ such that $(Vf)(x_0) > \varepsilon$ and that $(Vf)(x) \leqq (Vf)(x_0) - \varepsilon$ on the support $(f^+)$. Since $V(C_0(\Omega)^+)$ is dense in $X^+$ by hypothesis, there exists an $h \geqq 0$ such that $(Vh)(x_0) = (Vf)(x_0) - \varepsilon$ and $(Vh)(x) \geqq (Vf)(x_0) - \varepsilon$ on the support $(f^+)$. Hence $(Vf^+)(x) \leqq (Vf^-)(x) + (Vh)(x)$ on the support $(f^+)$, and so, by $(5)_1$, we must have $Vf \leqq Vh$. Thus we have a contradiction $(Vf)(x_0) \leqq (Vh)(x_0) = (Vf)(x_0) - \varepsilon$. We now turn to the general case $(Vf)(x_0) \geqq 0$, and take any compact set $\hat{E}$ of $\Omega$ containing $x_0$ as an interior point. Since $V(C_0(\Omega)^+)$ is dense in $X^+$, there exists a $g \in C_0(\Omega)^+$ such that $(Vg)(x_0) > \max_{x \in \Omega - \hat{E}} (Vg)(x)$. Then, for any $\varepsilon > 0$, the function $(V(f + \varepsilon g))(x)$ takes its positive maximum at a point $\in \hat{E}$ and not at points outside $\hat{E}$. Hence, as proved above, there must exist at least one point $y \in \hat{E}$ such that $f(y) + \varepsilon g(y) \geqq 0$. Therefore, we obtain $f(x_0) \geqq 0$ by letting $\varepsilon \downarrow 0$.

**Proof of Theorem 4.** By (8), we can define the operator $A$ through $AVf = -f$. Thus

(18) $$(\lambda I - A) Vf = \lambda Vf + f.$$

We first prove that the condition $V_\lambda f = 0$ with $\lambda > 0$ implies that $f = 0$. For, then $(V_\lambda f^+)(x) \leqq (Vf^-)(x)$ on the support $(f^+)$ and so $Vf^+ \leqq V_\lambda f^+ \leqq Vf^-$ by (5). Similarly we obtain $Vf^+ \geqq Vf^-$ and hence $Vf = 0$ so that $f = 0$ by (8).

Therefore the inverse $J_\lambda = (\lambda I - A)^{-1}$ exists as an operator which maps $(\lambda Vf + f)$ onto $Vf$. We can prove that

(19) $J_\lambda = (\lambda I - A)^{-1}$ is positive.

Let $h \geqq 0$ be $\in D(J_\lambda)$. Then $J_\lambda h = g = Vf$ with $f \in D(V)$ and

(20) $$h=(\lambda I-A)J_\lambda h=\lambda g-Ag=\lambda Vf+f.$$

Since $h\geqq 0$, we have $(\lambda Vf^{+})(x)\geqq(\lambda Vf^{-1})(x)+f^{-}(x)$ on the support $(f^{-})$ and so, by (5), $\lambda Vf^{+}\geqq\lambda Vf^{-}+f^{-}$, that is, $J_\lambda h=Vf\geqq\lambda^{-1}f^{-}\geqq 0$.

Since $A$ is a closed linear operator with $V$, we see that $D(J_\lambda)^a\supseteqq V_\lambda(C_0(\Omega))^a=X$ implies, by $V\geqq 0$, that $\lambda>0$ is in the resolvent set of $A$ and $J_\lambda=(\lambda I-A)^{-1}\in L(X, X)$.

We next show that (3) is true. Let $h\in V_\lambda(C_0(\Omega))$. Then, by (20), we can show that $\min_{x\in\Omega} h(x)\leqq(\lambda Vf)(x)\leqq\max_{x\in\Omega} h(x)$. In fact, let $(Vf)(x_0)=\max_{x\in\Omega}(Vf)(x)$. Then, by $(5)_2$, we have $f(x_0)\geqq 0$ so that $(\lambda Vf)(x)\leqq h(x_0)\leqq\max_{x\in\Omega} h(x)$. Similarly we obtain $(\lambda Vf)(x)\geqq\min_{x\in\Omega} h(x)$. Thus we have proved (3).

We finally prove that $Vf=s\text{-}\lim_{\mu\downarrow 0} J_\mu f$ for $f\in D(V)$. We have, by (18),

(21) $$Vf=\lambda J_\lambda Vf+J_\lambda f.$$

We also have, by $J_\lambda=(\lambda I-A)^{-1}$,

(22) $$(I-\lambda J_\lambda)f=-J_\lambda Af \quad \text{for } f\in D(A).$$

On the other hand, the range $R(A)=D(V)\supseteqq C_0(\Omega)$ is dense in $X$ and the range $R(J_\lambda)=D(A)=R(V)$ is dense in $X$. Thus we see that (22) implies that $R(I-\lambda J_\lambda)^a=X$. Hence, by (12)′, $s\text{-}\lim_{\lambda\downarrow 0}\lambda J_\lambda f=0$ for every $f\in X$. Therefore, by (21), we obtain $Vf=s\text{-}\lim J_\lambda f$ for every $f\in D(V)$.