27. Harmonic Summability of a Sequence of Fourier Coefficients

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1. Let $f(t)$ be a function which is integrable in the Lebesgue sense over the interval $(-\pi, \pi)$ and is defined outside this interval by periodicity. Let the Fourier series of $f(t)$ at $t=x$ be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{1}^{\infty}A_n(x). \tag{1.1}$$

Then the conjugate series of (1.1) is

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)=\sum_{1}^{\infty}B_n(x). \tag{1.2}$$

We write

$$\psi(t)=f(x+t)-f(x-t)-l. \tag{1.3}$$

Definition: The series $\sum a_n$ with the sequence of partial sum $\{S_n\}$ is said to be summable of harmonic means or summable $(H, 1)$, if

$$\operatorname*{Lt}_{n\to\infty}\frac{1}{\log n}\sum_{k=0}^{\infty}\frac{S_{n-k}}{k+1}=s, \tag{1.4}$$

where s is a definite number. When condition (1.4) is satisfied with S_n replaced by $\{S'_n\}$ then the above series will be said to be summable $(H, 1)$ (C, 1), where S'_n denotes the n-th Cesàro mean of order one of the sequence $\{S_n\}$.

The object of this paper is to prove the following

THEOREM: If

$$\Psi(t)=\int_0^t|\psi(u)|\,du=o(t)\quad\text{as } t\to 0, \tag{1.5}$$

$$\int_{\pi/n}^{\delta}\frac{|\psi(u+\pi/n)-\psi(u)|}{u}\log\frac{1}{u}du=o(\log n), \tag{1.6}$$

then the sequence $\{nB_n(x)\}$ is summable $(H, 1)$ $(C, 1)$ to the value l/π.

This theorem generalizes the result of Varshney [4].

2. We require the following lemmas:

Lemma 1. [3] For all values of n and x,

$$\sum_{r=1}^{n}\frac{\sin rt}{r}=o(1).$$

Lemma 2. [1] If $0<t<\pi$, Then

$$\sum_{r=1}^{n}\frac{\cos rt}{r}=o\left(\log\frac{1}{t}\right).$$

Lemma 3. [4] For $0\leqq t\leqq\pi/n$, we have

$$(\log n)^{-1}\sum_{r=1}^{n}\left(\frac{\sin rt}{r(n+1-r)t^2}-\frac{\cos rt}{(n+1-r)t}\right)=o(n).$$

3. Proof of the theorem: If we denote the $(C,1)$ transform of the sequence $\{nB_n(x)\}$ by σ_n, we know after Mohanty and Nanda [2]

$$\begin{aligned}\sigma_n-l/\pi&=\frac{1}{n}\sum_{r=1}^{n}rBr(x)-\frac{l}{\pi}\\&=\frac{1}{n}\int_0^\delta\psi(t)\left(\frac{\sin nt}{nt^2}-\frac{\cos nt}{t}\right)dt+o(1)\end{aligned}$$

by the Riemann-Lebesgue theorem, where δ is a fixed number.

On account of the regularity of harmonic summation, we need only prove that,

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{\pi}\int_0^\delta\psi(t)\left(\frac{\sin rt}{r(n+1-r)t^2}-\frac{\cos rt}{(n+1-r)t}\right)dt=o(1). \tag{3.1}$$

We get

$$\begin{aligned}&\int_0^\delta\psi(t)\frac{1}{\pi\log n}\sum_{r=1}^{n}\left(\frac{\sin rt}{r(n+1-r)t^2}-\frac{\cos rt}{(n+1-r)t}\right)dt\\&=\left(\int_0^{\pi/n}+\int_{\pi/n}^{\delta}\right)\psi(t)H_n(t)dt\\&=L_1+L_2,\quad\text{say.}\end{aligned}\tag{3.2}$$

Using (1.4) and Lemma 3, we have,

$$\begin{aligned}|L_1|&=\int_0^{\pi/n}|\psi(t)||H_n(t)|dt\\&=O\left(\int_0^{\pi/n}|\psi(t)|ndt\right)\\&=o(1).\end{aligned}\tag{3.3}$$

We set

$$\begin{aligned}L_2&=\frac{1}{\pi\log n}\int_{\pi/n}^{\delta}\frac{\psi(t)}{t}\sum_{r=1}^{n}\left(\frac{\sin rt}{r(n+1-r)t}-\frac{\cos rt}{(n+1-r)}\right)dt\\&=\frac{1}{\pi\log n}\int_{\pi/n}^{\delta}\frac{\psi(t)}{t}\sum_{r=1}^{n}\left(\frac{\sin rt}{(n+1)rt}+\frac{\sin rt}{(n+1)(n+1-r)t}-\frac{\cos rt}{(n+1-r)}\right)dt\\&=\frac{1}{\pi\log n}\int_{\pi/n}^{\delta}\frac{\psi(t)}{t}\left\{\sum_{r=1}^{n}\frac{\sin rt}{(n+1)rt}+\right.\\&\quad+\frac{1}{(n+1)}\sum_{r=1}^{n}\left(\frac{\sin(n+1)t\cos rt}{rt}-\frac{\cos(n+1)t\sin rt}{rt}\right)-\\&\quad\left.-\sum_{r=1}^{n}\left(\frac{\cos(n+1)t\cos rt}{r}+\frac{\sin(n+1)t\sin rt}{r}\right)\right\}dt\\&=L_{2,1}+L_{2,2}-L_{2,3}-L_{2,4}-L_{2,5},\quad\text{say.}\end{aligned}\tag{3.4}$$

With the help of Lemma 1 and the condition (1.4), we write

$$|L_{2,1}|=\frac{1}{(n+1)\pi\log n}\int_{\pi/n}^{\delta}\frac{|\psi(t)|}{t^2}\left|\sum_{r=1}^{n}\frac{\sin rt}{r}\right|dt$$

$$=O\left(\frac{1}{(n+1)\log n}\int_{\pi/n}^{\delta}\frac{|\psi(t)|}{t^2}dt\right)$$

$$=O\left(\frac{1}{(n+1)\log n}\left[\frac{\Psi(t)}{t^2}\right]_{\pi/n}^{\delta}+\frac{2}{(n+1)\log n}\int_{\pi/n}^{\delta}\frac{\Psi(t)}{t^3}dt\right)$$

$$=o(1)+o\left(\frac{1}{(n+1)\log n}\left[\frac{1}{t}\right]_{\pi/n}^{\delta}\right)$$

$$=o(1) \quad \text{as } n\to\infty. \tag{3.5}$$

Similarly,

$$|L_{2,3}|=o(1), \quad \text{as } n\to\infty. \tag{3.6}$$

Now we shall evalute $L_{2,2}$. Using Lemma 2, we have,

$$L_{2,2}=\frac{1}{(n+1)\pi\log n}\int_{\pi/n}^{\delta}\frac{\psi(t)}{t^2}\sin(n+1)t\sum_{r=1}^{n}\frac{\cos rt}{r}dt$$

$$=O\left(\frac{1}{(n+1)\log n}\int_{\pi/n}^{\delta}\frac{\psi(t)}{t^2}\sin nt\log\frac{1}{t}dt\right)+o(1)$$

$$=O\left(\frac{1}{\log n}\int_{\pi/n}^{\delta}\frac{\psi(t)}{t}\sin nt\log\frac{1}{t}dt\right)+o(1),$$

by the regularity of the method of summation,

$$=O\left(-\frac{1}{\log n}\int_{0}^{\delta-\pi/n}\frac{\psi(t+\pi/n)}{t+\pi/n}\sin nt\log\frac{1}{t+\pi/n}dt\right)+o(1)$$

$$=O\left\{\frac{1}{2\log n}\int_{\pi/n}^{\delta}\frac{\psi(t)}{t}\sin nt\log\frac{1}{t}dt-\right.$$

$$\left.-\frac{1}{2\log n}\int_{0}^{\delta-\pi/n}\frac{\psi(t+\pi/n)}{t+\pi/n}\sin nt\log\frac{1}{t+\pi/n}dt\right\}+o(1)$$

$$=O\left\{\frac{1}{\log n}\left(\int_{\pi/n}^{\delta-\pi/n}+\int_{\delta-\pi/n}^{\delta}\right)\frac{\psi(t)}{t}\sin nt\log\frac{1}{t}dt-\right.$$

$$\left.-\frac{1}{\log n}\left(\int_{0}^{\pi/n}+\int_{\pi/n}^{\delta-\pi/n}\right)\frac{\psi(t+\pi/n)}{t+\pi/n}\sin nt\log\frac{1}{t+\pi/n}dt\right\}+o(1)$$

$$=O\left\{\frac{1}{\log n}\int_{\pi/n}^{\delta-\pi/n}\left[\frac{\psi(t)}{t}\log\frac{1}{t}-\frac{\psi(t+\pi/n)}{t+\pi/n}\log\frac{1}{t+\pi/n}\right]\sin ntdt\right\}-$$

$$-O\left(\frac{1}{\log n}\int_{0}^{\pi/n}\frac{\psi(t+\pi/n)}{t+\pi/n}\sin nt\log\frac{1}{t+\pi/n}dt\right)+$$

$$+O\left(\frac{1}{\log n}\int_{\delta-\pi/n}^{\delta}\frac{\psi(t)}{t}\sin nt\log\frac{1}{t}dt\right)+o(1)$$

$$=O\left[\frac{1}{\log n}\int_{\pi/n}^{\delta-\pi/n}\left\{\left(\frac{\psi(t)}{t}\log\frac{1}{t}-\frac{\psi(t+\pi/n)}{t}\log\frac{1}{t}\right)+\right.\right.$$

$$+\left(\frac{\psi(t+\pi/n)}{t}\log\frac{1}{t}-\frac{\psi(t+\pi/n)}{t}\log\frac{1}{t+\pi/n}\right)+$$

$$+\left(\frac{\psi(t+\pi/n)}{t}\log\frac{1}{t+\pi/n}-\frac{\psi(t+\pi/n)}{t+\pi/n}\log\frac{1}{t+\pi/n}\right)\Big\}\sin ntdt\Big]-$$
$$-O\left(\frac{1}{\log n}\int_0^{\pi/n}\frac{\psi(t+\pi/n)}{t+\pi/n}\sin nt\log\frac{1}{t+\pi/n}dt\right)+$$
$$+\left(\frac{1}{\log n}\int_{\delta-\pi/n}^{\delta}\frac{\psi(t)}{t}\sin nt\log\frac{1}{t}dt\right)+o(1).$$

(3.7)$=(P_1+P_2+P_3)-P_4+P_5$, say.

With the hypothesis (1.6), we get

$$|P_1|=o(1). \tag{3.8}$$

Further,

$$|P_2|=O\left(\frac{1}{\log n}\int_{\pi/n}^{\delta}\frac{|\psi(t+\pi/n)|}{t}\Big\{\log\frac{1}{t}-\log\frac{1}{t+\pi/n}\Big\}dt\right)$$
$$=O\left(\frac{1}{\log n}\int_{\pi/n}^{\delta}\frac{|\psi(t+\pi/n)|}{t}o\left(\frac{1}{nt}\right)dt\right)$$
$$=O\left(\frac{1}{n\log n}\int_{\pi/n}^{\delta}\frac{|\psi(t+\pi/n)|}{t^2}dt\right)$$
$$=o(1)\quad\text{by (3.5)}. \tag{3.9}$$

Now,

$$|P_3|=O\left(\frac{1}{n\log n}\int_{\pi/n}^{\delta}\frac{|\psi(t+\pi/n)|}{t(t+\pi/n)}\log\frac{1}{t+\pi/n}|\sin nt|\,dt\right)$$
$$=O\left(\frac{1}{n}\int_{\pi/n}^{\delta}\frac{|\psi(t+\pi/n)|}{t^2}dt\right)$$
$$=o(1)\quad\text{by (3.5)}. \tag{3.10}$$

Again,

$$P_4=O\left(\frac{1}{\log n}\int_0^{\pi/n}\frac{\psi(t+\pi/n)}{t+\pi/n}\sin nt\log\frac{1}{t+\pi/n}dt\right)$$
$$=O\left(\frac{1}{\log n}\int_{\pi/n}^{2\pi/n}\frac{\psi(t)}{t}\sin nt\log\frac{1}{t}dt\right)$$
$$=O\left(\frac{1}{\log n}\int_{\pi/n}^{2\pi/n}\frac{\psi(t)}{t}o(nt)\log\frac{1}{t}dt\right).$$

Thus,

$$|P_4|=O\left(n\int_{\pi/n}^{2\pi/n}|\psi(t)|\,dt\right)$$
$$=O(n)o\left(\frac{1}{n}\right)\quad\text{by (1.5)},$$
$$=o(1). \tag{3.11}$$

Also,

$$|P_5|=O\left(\frac{1}{\log n}\int_{\delta-\pi/n}^{\delta}\frac{|\psi(t)|}{t}|\sin nt|\log\frac{1}{t}dt\right)$$
$$=o(1), \tag{3.12}$$

by the regularity of the method of summation, since the interval

$(\delta-\pi/n, \delta)$ tends to zero as $n\to\infty$.

Consequently from (3.8), (3.9), (3.10), (3.11), and (3.12), we have

$$|L_{2,2}|=o(1) \quad \text{as } n\to\infty. \tag{3.13}$$

Similarly

$$|L_{2,4}|=o(1) \quad \text{as } n\to\infty. \tag{3.14}$$

With the help of Lemma 1 and condition (1.5) we get

$$\begin{aligned}|L_{2,5}| &= O\left(\frac{1}{\log n}\int_{\pi/n}^{\delta}\frac{|\psi(t)|}{t}\,|\operatorname{Sin}(n+1)t|\left|\sum_{r=1}^{n}\frac{\sin rt}{r}\right|dt\right)\\ &= O\left(\frac{1}{\log n}\int_{\pi/n}^{\delta}\frac{|\psi(t)|}{t}dt\right)\end{aligned}$$

$$=o(1), \text{ as we proved in } L_{2,1}. \tag{3.15}$$

Collection of (3.3), (3.5), (3.6), (3.14), (3.15), and (3.13) completes the proof of the theorem.

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