64. On the Conditional Expectation of a Partial Isometry in a Certain von Neumann Algebra

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1. In 1943, J. von Neumann stated in his monumental paper [3; Lemma 5.2.3] that the crossed product of an abelian von Neumann algebra with a faithful normal trace by an ergodic abelian group of trace preserving automorphisms is a continuous hyperfinite factor. He wrote that the lemma required some rather deep analysis on the decomposition of measure preserving transformations and the proof would be published in a separate paper. Since then no proof is avairable to us.

Twenty years after, H. A. Dye published an interesting paper [2] on a deep analysis of measure preserving transformations and established a theorem [2; Cor. 6.1] which implies the above cited lemma of von Neumann. It seems that Dye's success is an important advance in the recent theory of von Neumann algebras.

However, a slight defect is admitted in a part of Dye's proof. His proof depends principally upon [2; Lemma 6.1]: Let $\mathcal{A}$ be a regular maximal abelian self-adjoint subalgebra of a von Neumann algebra $\mathcal{B}$ of finite type with the faithful normal trace, and let C be an intermediate von Neumann subalgebra between $\mathcal{A}$ and $\mathcal{B}$. If U is a unitary operator of $\mathcal{B}$ which preserves $\mathcal{A}$ in the sense that $U\mathcal{A}U^*\subseteqq\mathcal{A}$, then the conditional expectation $U^E=E[U\mid C]$ of U conditioned by C in the sense of H. Umegaki [4] is a partial isometry of $\mathcal{B}$. To prove this, he stated at [2; (6.4)] that

$$(*)\qquad UAU^*=VAV^*,$$

for all $A\in\mathcal{A}$, where V is a partial isometry belonging to the polar decomposition of U^E, that is,

$$U^E=E[U\mid C]=V[U^{E*}U^E]^{\frac{1}{2}}.$$

Unfortunately, (*) is not true in general.

But, the essential part of the proof of Dye will be salvaged by the following

THEOREM. *If $\mathcal{A}$ is a maximal abelian subalgebra of a von Neumann algebra $\mathcal{B}$ with a faithful finite normal trace, if C is an intermediate von Neumann subalgebra between $\mathcal{A}$ and $\mathcal{B}$, and if U is a partial isometry of $\mathcal{B}$ preserving $\mathcal{A}$ in the sense that*

$$U\mathcal{A}U^*\subseteqq\mathcal{A} \quad and \quad U^*\mathcal{A}U\subseteqq\mathcal{A},$$

then the conditional expectation $U^E=E[U|C]$ *conditioned by* C *is a partial isometry of* C *preserving* $\mathcal{A}$.

2. According to the investigation [4] of Umegaki, the conditional expectation $A\rightarrow A^E$ is a positive linear adjoint preserving mapping which satisfies

(1) $(BC)^E=B^EC$, for $B\in\mathcal{B}$ and $C\in C$.

By (1) and the fact that $\mathcal{A}\subseteqq C$, we have

(2) $U^EA=(UA)^E=(UU^*UA)^E=(UAU^*U)^E=UAU^*U^E$,

for all $A\in\mathcal{A}$. Let S be the absolute of U^E, i.e., $S=[U^{E*}U^E]^{\frac{1}{2}}$. By (2) and $U^{*E}=U^{E*}$, we have $S^2A=AS^2$ for all $A\in\mathcal{A}$, and so $S^2\in\mathcal{A}$ by the maximal abelian character of $\mathcal{A}$. Consequently, we have $S\in\mathcal{A}$. Let

(3) $U^E=VS$

be the polar decomposition of U^E. Substitute VS for U^E in (2), we have $UAU^*VS=VSA$. Then

$$AU^*VS=AU^*UUVS=U^*UAU^*VS=U^*VSA,$$

for all $A\in\mathcal{A}$. Therefore U^*VS belongs to $\mathcal{A}$ since $\mathcal{A}$ is maximally abelian, and so by (3)

$$U^*VS=(U^*VS)^E=(U^*U^E)^E=U^{*E}U^E=S^2.$$

Hence,

$$S^4=SV^*UU^*VS=U^{*E}UU^*VS=(U^*UU^*)^EU^E=U^{*E}U^E=S^2,$$

which shows that S^2 is a projection. Therefore S is a projection since S is non-negative. Because V^*V is the support of S, we have $S=V^*V$ and

$$U^E=VS=VV^*V=V.$$

This proves the first half of the theorem.

3. Let us now return to the lemma of Dye. By the theorem, U^E is a partial isometry when U is unitary. Let us put $E=V^*V$ and $F=VV^*$. If we substitute V for U^E in (2), then we have $VA=UAU^*V$, whence we have

$$VAV^*=UAU^*VV^*=UAU^*F.$$

Therefore, (*) must be replaced by

(**) $(UAU^*)F=VAV^*$,

for all $A\in\mathcal{A}$. Since $A\rightarrow UAU^*$ is an automorphism of $\mathcal{A}$ and F is a projection of $\mathcal{A}$, we have

$$V\mathcal{A}V^*=\mathcal{A}F\subseteqq\mathcal{A}.$$

By the same argument applied to U^*, we have also $V^*\mathcal{A}V\subseteqq\mathcal{A}$. Therefore the conditional expectation V of the given unitary U conditioned by an intermediate subalgebra C preserves $\mathcal{A}$ and the essential part of the lemma of Dye is obtained. The proof of the second part of the theorem is completely analogous.

4. Finally, let us observe an another consequence of the theorem. If $\mathcal{D}$ is an another intermediate von Neumann subalgebra between $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{D}\subseteqq C$. Then we have by a property of the conditional expectation due to Umegaki [5]

$$E[U\,|\,\mathcal{D}]=E[E[U\,|\,C]\,|\,\mathcal{D}],$$

and $E[U\,|\,\mathcal{D}]$ is a partial isometry of $\mathcal{D}$ by the theorem. Hence we can conclude, using the definition of martingales introduced by Umegaki [5] as M-nets, that *the family* $\{E[U\,|\,C];\mathcal{A}\subseteqq C\subseteqq\mathcal{B}\}$ *of partial isometries forms an example of non-commutative simple martingale which terminates at* $E[U\,|\,\mathcal{A}]$. The character of the convergence of the martingale is decidable by the theorems of Umegaki [5; Th. 2 and Th. 3].

References

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