114. A Note on Elliptic Differential Operators of the Second Order

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§ 0. **Introduction.** Let $x=(x_1, \cdots, x_m)$ donote a variable point in the euclidean m-space E_m and ∂_i, ∂_{ij} denote partial differentiations $\partial/\partial x_i$ and $\partial^2/\partial x_i\partial x_j$ respectively. Let L be an elliptic differential operator of the second order with real coefficients:

$$Lu=\sum_{ij=1,1}^{m,m} a_{ij}(x)\partial_{ij}u+\sum_{i=1}^{m} b_i(x)\partial_i u$$

defined for $u\in C^2(G)$, where G is an open domain in E_m, a_{ij}, $b_i\in C(G)$ and the condition of ellipticity

$$A^{-1}\,|\,\xi\,|^2\leqq\sum_{ij=1,1}^{m,m} a_{ij}(x)\xi_i\xi_j\leqq A\,|\,\xi\,|^2$$

for all $\xi\in R^m$ and $x\in G$ holds with a fixed constant $A\geqq 1$.

K. Akô [1] gave an extension of the domain of the operator L as follows:

A function $u\in C(G)$ is said to be in the domain of $\mathcal{L}$ and satisfy the equation $\mathcal{L}u=f$ in G if the following conditions are satisfied

i) $f\in C(G)$.

ii) *For each* $x^0\in G$ *there exist a neighborhood* U *of* x^0 *and sequences*

$\{u_n\}_{n=1}^{\infty}\subset C^2(U)$, $\{a_{ij}^{(n)}\}_{n=1}^{\infty}\subset C(U)$, $\{b_i^{(n)}\}_{n=1}^{\infty}\subset C(U)$, *and* $\{f_n\}_{n=1}^{\infty}\subset C(U)$ *such that*

$$u_n\rightrightarrows u,\ a_{i,j}^{(n)}\rightrightarrows a_{ij},\ b_i^{(n)}\rightrightarrows b_i$$

and

$$L^{(n)}u_n=f_n\rightrightarrows f^{1)}\ \text{in } U \text{ as } n\to\infty,$$

where

$$L^{(n)}=\sum_{i,j=1,1}^{m,m} a_{ij}^{(n)}(x)\partial_{ij}+\sum_{i=1}^{m} b_i^{(n)}\partial_i.$$

But as Akô has not proved the uniqueness of the extension, we shall here give the proof for the uniqueness.

In § 1 we shall consider the case where the sequence of the approximating operators $\{L^{(n)}\}$ is given previously independent of $\{u_n\}$, and give an elementary proof by Paraf's principle.

In § 2 we shall consider the general case from the viewpoint of the functional space (L^2). Then the proof will be at hand by virture of the recent theory of partial differential operators even for elliptic operators of arbitrarily high orders. We shall here, however, prefer more elementary way using a device of H. O. Cordes.

§ 1. **An elementary method.** In this section, for the sequence

1) The symbol $\rightrightarrows$ denotes the uniform convergence.

$\{L^{(n)}\}$ previously given independent of $\{u_n\}$, we shall prove the uniqueness of the extension of the operator L as elementary as possible. We use the same notations as before.

Theorem 1. *Let the sequences* $\{u_n^{(k)}\}_{n=1}^{\infty}$ $(k=1,2)$ *from* $C^2(U)$ *be such that* $u_n^{(k)} \rightrightarrows u$ *and* $L^{(n)}u_n^{(k)}=f_n^{(k)} \rightrightarrows f^{(k)}$ $(k=1,2)$ *in* U *as* $n\to\infty$, *where* U *is an open subdomain of* G. *Then we have* $f^{(1)}(x)=f^{(2)}(x)$ *in* U.

Proof: Setting $w_n=u_n^{(1)}-u_n^{(2)}$, $h_n=f_n^{(1)}-f_n^{(2)}$, and $h=f^{(1)}-f^{(2)}$, we have to show $h(x)=0$ for $x\in U$. If we had $h(x^0)\neq 0$ for some $x^0\in U$ we should derive a contradiction.

We can assume there exist constants $\alpha>0, \delta>0, \sigma>0$, and B such that

$$h_n(x)\geqq\alpha,\ \sum_{i=1}^m a_{ii}^{(n)}(x)\leqq\sigma$$

and

$$|b^{(n)}(x)|=(\sum_{i=1}^m b_i^{(n)}(x)^2)^{1/2}\leqq B$$

for $x\in U_\delta(x^0)=\{x;\ |x-x^0|\leqq\delta\}$ and all n.

Setting $\varphi_n(x)=\varepsilon_n-(4\sigma)^{-1}\alpha(\delta^2-|x-x^0|^2)$ with $\varepsilon_n=\underset{|x-x^0|=\delta}{\mathrm{Max}}\ W_n(x)$, we have, assuming $\delta<\sigma B^{-1}$,

$$L^{(n)}\varphi_n<\alpha\leqq h_n(x)=L^{(n)}w_n \quad \text{for} \quad x\in U_\delta(x^0)$$

and $\varphi_n(x)-w_n(x)\geqq 0$ if $|x-x^0|=\delta$. Thus, as φ_n-w_n cannot be minimum in $|x-x^0|<\delta$, we have

$$\varphi_n(x)>w_n(x) \quad \text{for} \quad |x-x^0|<\delta.$$

Then, letting $n\to\infty$, as $w_n\rightrightarrows 0$ in U, we get the contradiction

$$-(4\sigma)^{-1}\alpha(\delta^2-|x-x^0|^2)\geqq 0 \quad \text{for} \quad |x-x^0|<\delta. \qquad \text{Q.E.D.}$$

Now for the special case $L^{(n)}=L$ for all n, let $\bar{L}$ be the extension of the operator L. Then we have:

Theorem 2. *The extension* $\mathcal{L}$ *of* L *is uniquely determined if it is restricted to the domain of* $\bar{L}$.

Proof: Let u belong to the domain of $\bar{L}$ and $\bar{L}u=f$. Then for every $x^0\in G$ there exist a neighborhood U of x^0 and sequences $\{u_n\}_{n=1}^{\infty}\subset C^2(U)$ and $\{f_n\}_{n=1}^{\infty}\subset C(U)$ such that $u_n\rightrightarrows u$ on U and $Lu_n=f_n\rightrightarrows f$ on U. Further let $\{u_n'\}_{n=1}^{\infty}\subset C^2(U)$ and $\{f_n'\}_{n=1}^{\infty}\subset C(U)$ be such sequences that, with $L^{(n)}$ given in § 0,

$$u_n'\rightrightarrows u \text{ and } L^{(n)}u_n'=f_n'\rightrightarrows f' \text{ on } U.$$

We have then to show $f'(x)=f(x)$ for $x\in U$. As we can assume that the closure $\bar{U}$ of U is in G and compact, and

$$|Lu_n(x)-L^{(\nu)}u_n(x)|\leqq\varepsilon_\nu \underset{i,j}{\mathrm{Max}}\ \underset{x\in\bar{U}}{\mathrm{Max}}\ |\partial_{ij}u_n(x)|+\varepsilon_\nu' \underset{i}{\mathrm{Max}}\ \underset{x\in\bar{U}}{\mathrm{Max}}\ |\partial_i u_n(x)|,$$

where

$$\varepsilon_\nu=\sum_{ij=1,1}^{m,m}\underset{x\in\bar{U}}{\mathrm{Max}}\ |a_{ij}^{(\nu)}(x)-a_{ij}(x)|,\ \varepsilon_\nu'=\sum_{i=1}^m \underset{x\in\bar{U}}{\mathrm{Max}}\ |b_i^{(\nu)}(x)-b_i(x)|$$

with $\varepsilon_\nu\to 0$, $\varepsilon_\nu'\to 0$ for $\nu\to\infty$, we can get a sequence $\{\nu(n)\}$ of natural numbers such that $\nu(n)\geqq n$ and

$$|Lu_n(x)-L^{(\nu(n))}u_n(x)|\rightrightarrows 0 \text{ on } U.$$

Then, setting $L^{(\nu(n))}u_n=f_n''$ we have $u_n\rightrightarrows u$, $u'_{\nu(n)}\rightrightarrows u$, and $f_n''\rightrightarrows f$ on U, but $L^{(\nu(n))}u'_{\nu(n)}=f'_{\nu(n)}\rightrightarrows f'$ on U. Thus by Theorem 1, we obtain $f'(x)=f(x)$ on U. Q.E.D.

§2. Consideration in (L^2). To consider our problem in the functional space (L^2) in an elementary way, we use the following:

Lemma. *Let $U_\delta(x^0)=\{x;\,|x-x^0|\leqq\delta\}\subset G$ and define the function $S_\delta(x)$ by $S_\delta(x)=(\delta^2-|x-x^0|^2)$ if $|x-x^0|\leqq\delta$, $S_\delta(x)=0$ if $|x-x^0|>\delta$. Then, for any constant $\kappa>0$, we have for every $u\in C^2(G)$*

$$(1)\qquad \int_{U_\delta}\sum_{i=1}^m(\partial_i u)^2S_\delta^2dx\leqq\kappa\delta^{-2}\int_{U_\delta}(\Delta u)^2S_\delta^4dx+C_\kappa\delta^2\int_{U_\delta}u^2dx \text{ [2)]}$$

with a constant C_κ depending only on κ, and

$$(2)\qquad \int_{U_\delta}\sum_{i,j=1,1}^{m,m}(\partial_{ij}u)^2S_\delta^4dx\leqq(1+\kappa)\int_{U_\delta}(\Delta u)^2S_\delta^4dx+C_{m,\kappa}\delta^4\int_{U_\delta}u^2dx$$

with a constant $C_{m,\kappa}$ depending only on m and κ.

Proof: Omitted. Cf. Cordes [2].

Now let $\{L^{(n)}\}_{n=1}$ be a sequence of operators defined for $C^2(U)$ $(U\subset G)$ with the same conditions as in §0.

Theorem 3. *Let $\{u_n\}_{n=1}^\infty$ be a sequence from $C^2(U)$ such that both sequences $\{u_n\}_{n=1}^\infty$ and $\{L^{(n)}u_n\}_{n=1}^\infty$ are convergent in $L^2(U)$ (mean convergent). Then the sequences $\{\partial_{ij}u_n\}_{n=1}^\infty$ and $\{\partial_i u_n\}_{n=1}^\infty$ $(i,j=1,\cdots,m)$, restricted on any compact part K of U, are convergent in $L^2(K)$.*

Proof: First let $x^0\in K$. Since the properties of the functional spaces $C^k(k=0,1,2)$ and L^2 are not injured by any non-singular linear transformation of the independent variables in E_m, we can assume, for the coefficients of the principal part of L at $x=x^0$, that

$$(4)\qquad a_{ij}(x^0)=\delta_{ij}\quad\text{(Kronecker's delta)}.$$

Then, there exist constants $\delta>0$ and $\beta>0$ and a natural number N such that

$$(5)\qquad \begin{cases}\sum_{i,j=1,1}^{m,m}(a_{ij}^{(n)}(x)-\delta_{ij})^2<1/2,\\ \sum_{i=1}^m b_i^{(n)}(x)^2\leqq\beta\end{cases}$$

for $x\in U_\delta(x^0)$ and $n\geqq N$. Thus, as for any constant $\kappa>0$ by (5)

$$\begin{aligned}(\Delta u)^2&\leqq(1+\kappa^{-1})(L^{(n)}u)^2+(1+\kappa)((L^{(n)}-\Delta)u)^2\\ &\leqq(1+\kappa^{-1})(L^{(n)}u)^2+2^{-1}(1+\kappa)^2\sum_{i,j=1,1}^{m,m}(\partial_{ij}u)^2\\ &\quad+\beta(1+\kappa^{-1})(1+\kappa)\sum_{i=1}^m(\partial_i u)^2,\end{aligned}$$

we have from (1), replacing κ by $(2\beta)^{-1}\kappa^2(1+\kappa)^2$, and (2) of Lemma, with a constant $C'_{m,\kappa,\beta}$ depending only on m, κ, and β

$$\begin{aligned}\int_{U_\delta}(\Delta u)^2\cdot S_\delta^4dx&\leqq(1+\kappa^{-1})\int_{U_\delta}(L^{(n)}u)^2\cdot S_\delta^4dx\\ &\quad+2^{-1}(1+\kappa)^4\int_{U_\delta}(\Delta u)^2\cdot S_\delta^4dx+C'_{m,\kappa,\beta}\delta^4\int_{U_\delta}u^2dx\,.\end{aligned}$$

2) Δ means the laplacian operator $\Delta=\sum_{i=1}^m\partial_{ii}$.

Hence, choosing κ in such a way that $2^{-1}(1+\kappa)^4 \leqq 3/4$, for example $\kappa=(10)^{-1}$, we get

$$\int_{U_\delta}(\Delta u)^2 \cdot S_\delta^4 dx \leqq 44\int_{U_\delta}(L^{(n)}u)^2 \cdot S_\delta^4 dx + C'_{m,\beta}\cdot\delta^4\int_{U_\delta} u^2 dx$$

with a constant $C'_{m,\beta}$ depending only on m and β. Thus, by (2) of Lemma

$$(6) \qquad \int_{U_\delta} \sum_{i,j=1,1}^{m,m} (\partial_{ij}u)^2 \cdot S_\delta^4 dx \leqq 50\int_{U_\delta}(L^{(n)}u)^2 \cdot S_\delta^4 dx + C_{m,\beta}\delta^4\int_{U_\delta} u^2 dx$$

with a constant $C_{m,\beta}$ depending only on m and β.

Now from (6) we can easily see that, without the condition (4), for each $x^0 \in K$ there exist neighborhoods V and W of x^0 such that $\bar{V} \subset W \subset \bar{W} \subset U$ and for every $u \in C^2(U)$ holds

$$\int_V \sum_{i,j=1,1}^{m,m} (\partial_{ij}u)^2 \cdot dx \leqq C_{V,W}\int_W (L^{(n)}u)^2 dx + C'_{V,W}\int_W u^2 dx$$

with constants $C_{V,W}$ and $C'_{V,W}$ depending only on L, V, and W, if $n \geqq N(x^0)$. Thus, applying Borel's convering theorem on the compact set $K \subset U$ we obtain, if $n \geqq N(K, U, L)$,

$$(7) \qquad \int_K \sum_{i,j=1,1}^{m,m} (\partial_{ij}u)^2 dx \leqq C_{K,U,L}\int_U (L^{(n)}u)^2 dx + C'_{K,U,L}\int_U u^2 dx$$

for every $u \in C^2(U)$ with constants $C_{K,U,L}$ and $C'_{K,U,L}$ depending only on K, U, and L.

Again with the condition (4), from (1) and (6) we get, as $(\Delta u)^2 \leqq m\sum_{i,j=1,1}^{m,m}(\partial_{ij}u)^2$, setting $\kappa=(50\,\mathrm{m})^{-1}$,

$$\int_{U_\delta} \sum_{i=1}^{m} (\partial_i u)^2 S_\delta^2 dx \leqq \delta^{-2}\int_{U_\delta}(L^{(n)}u)^2 \cdot S_\delta^4 dx + C'_{m,\beta}\delta^2\int_{U_\delta} u^2 dx.$$

Hence by the similar argument as above we obtain, if $n \geqq N'(K, U, L)$,

$$(8) \qquad \int_K \sum_{i=1}^{m} (\partial_i u)^2 dx \leqq C^1_{K,U,L}\int_U (L^{(n)}u)^2 dx + C^2_{K,U,L}\int_U u^2 dx$$

for every $u \in C^2(U)$ with constants $C^1_{K,U,L}$ and $C^2_{K,U,L}$ depending only on K, U, and L.

Now let V be a compact neighborhood of K such that $K \subset V^0$[3] $\subset V \subset U$. Then, replacing K by V in (7) and (8), we see that the sequences $\{\partial_{ij}u_n\}_{n=1}^\infty$ and $\{\partial_i u_n\}_{n=1}^\infty$, restricted on V, are bounded in $L^2(V)$. Then with $f_n = L^{(n)}u_n$, as

$$L(u_n - u_\nu) = (L^{(\nu)} - L)u_\nu - (L^{(n)} - L)u_n + f_n - f_\nu$$

and $(L^{(\nu)} - L)u_\nu \to 0$ in $L^2(V)$, $(L^{(n)} - L)u_n \to 0$ in $L^2(V)$ and $f_n - f_\nu \to 0$ in $L^2(V)$ as $n, \nu \to \infty$, we have

$$L(u_n - u_\nu) \to 0 \text{ in } L^2(V) \text{ as } n, \nu \to \infty.$$

Therefore, by (7) and (8), where $L^{(n)} = L$ for all n, we obtain that $\{\partial_{ij}u_n\}_{n=1}^\infty$ and $\{\partial_i u_n\}_{n=1}^\infty$ are convergent in $L^2(K)$. Q.E.D.

Last we have the desired:

Theorem 4. *The extension $\mathcal{L}$ of the operator L, given in*

3) V^0 denotes the open kernel of V.

§ 0, *is uniquely determined. Namely, let* $\{u_n^{(\alpha)}\}_{n=1}^{\infty}\subset C^2(U)$ *and* $\{f_n^{(\alpha)}\}_{n=1}^{\infty}\subset C(U)$ $(\alpha=1, 2)$ *be sequences such that*

$$L^{(n)}u_n^{(1)}=f_n^{(1)},\ \acute{L}^{(n)}u_n^{(2)}=f_n^{(2)},\ u_n^{(\alpha)}\rightrightarrows u \quad \text{and} \quad f_n^{(\alpha)}\rightrightarrows f^{(\alpha)}$$

on U, *where* $L^{(n)}=\sum_{i,j=1,1}^{m,m} a_{ij}^{(n)}(x)\partial_{ij}+\sum_{i=1}^{m} b_i^{(n)}(x)\partial_i$ *and*

$$\acute{L}^{(n)}=\sum_{i,j=1,1}^{m,m} \acute{a}_{ij}^{(n)}(x)\partial_{ij}+\sum_{i=1}^{m} \acute{b}_i^{(n)}(x)\partial_i$$

with $a_{ij}^{(n)}\rightrightarrows a_{ij}$, $\acute{a}_{ij}^{(n)}\rightrightarrows a_{ij}$, $b_i^{(n)}\rightrightarrows b_i$, *and* $\acute{b}_i^{(n)}\rightrightarrows b_i$ *on* U. *Then we have* $f^{(1)}(x)=f^{(2)}(x)$ *on* U.

Proof: By Theorem 3, assuming U to be bounded, on any compact part K of U, the sequences $\{\partial_{ij}u_n^{(\alpha)}\}_{n=1}^{\infty}$ and $\{\partial_i u_n^{(\alpha)}\}_{n=1}^{\infty}$ $(i, j=1, \cdots, m)$ $(\alpha=1, 2)$ are convergent in $L^2(K)$. Hence $\{\partial_{ij}(u_n^{(1)}-u_n^{(2)})\}_{n=1}^{\infty}$ and $\{\partial_i(u_n^{(1)}-u_n^{(2)})\}_{n=1}^{\infty}$ are convergent in $L^2(K)$. But, as $(u_n^{(1)}-u_n^{(2)})\longrightarrow 0$ in $L^2(U)$, we have, for any $\varphi\in C^2(U)$ with compact support in U,

$$\int_U \partial_{ij}(u_n^{(1)}-u_n^{(2)})\cdot\varphi dx=\int_U (u_n^{(1)}-u_n^{(2)})\cdot\partial_{ij}\varphi dx\longrightarrow 0$$

and

$$\int_U \partial_i(u_n^{(1)}-u_n^{(2)})\cdot\varphi dx=-\int_U (u_n^{(1)}-u_n^{(2)})\cdot\partial_i\varphi dx\longrightarrow 0.$$

Then $\partial_{ij}(u_n^{(1)}-u_n^{(2)})\longrightarrow 0$ and $\partial_i(u_n^{(1)}-u_n^{(2)})\longrightarrow 0$ in $L^2(K)$. Therefore $\{L^{(n)}u_n^{(1)}\}_{n=1}^{\infty}$ and $\{\acute{L}^{(n)}u_n^{(2)}\}_{n=1}^{\infty}$ must converge to the same function in $L^2(K)$. But, as $f^{(\alpha)}(x)$ $(\alpha=1, 2)$ are continuous on U, we have $f^{(1)}(x)=f^{(2)}(x)$ on U. Q.E.D.

References

[1] K. Akô: Subfunctions for quasilinear elliptic differential equations. J. Fac. Sci. Univ. Tokyo, Sect. I, **9**, 403-416 (1963).

[2] H.O. Cordes: Vereinfachter Beweis der Existenz einer Apriori-Hölderkonstante. Math. Ann., **138**, 155-178 (1959).