142. On the Total Regularity of Riemann Summability

By Hiroshi HIROKAWA
Department of Mathematics, Chiba University, Chiba, Japan

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§ 1. A method of summation is said to be regular if it assigns to every convergent series its actual value. If it furthermore assigns the value $+\infty$ to every series which diverges to $+\infty$, it is said to be totally regular. In this paper we shall consider the total regularity of Riemann summability. Throughout this paper, p denotes a positive integer. A series $\sum_{n=1}^{\infty} a_n$ is said to be summable (R, p) to s if the series in

$$f_p(t)=\sum_{n=1}^{\infty} a_n \left(\frac{\sin nt}{nt}\right)^p$$

converges in some interval $0<t<t_0$ and $f_p(t)\to s$ as $t\to 0+$. A series $\sum_{n=1}^{\infty} a_n$, with its partial sum s_n, is said to be summable (R_p) to s if the series in

$$F_p(t)=C_p^{-1}t\sum_{n=1}^{\infty} s_n \left(\frac{\sin nt}{nt}\right)^p.$$

where

$$C_p=\int_0^{\infty} u^{-p}\sin^p u\, du,$$

converges in some interval $0<t<t_0$ and $F_p(t)\to s$ as $t\to 0+$. It is well-known that the methods (R, p) and (R_p) are regular when $p\geqq 2$, while the methods $(R, 1)$ and (R_1) are not regular. (See, for example, [2]). But, concerning the total regularity of Riemman summability, the S.C. Lee's result [4] seems to be the only one. He proved that the method $(R, 2)$ is not totally regular.

§ 2. We shall first prove the following theorem.

THEOREM 1. The method (R, p) is not totally regular when $p\geqq 2$. More precisely, given a monotone increasing sequence $\{W_n\}$ tending to $+\infty$ such that $W_n\ n^{-p}\to 0$ as $n\to\infty$, there exists a series $\sum_{n=1}^{\infty} a_n$ with $|a_n|\leqq 2W_n/n$ for all n, such that

$$\sum_{n=1}^{\infty} a_n=+\infty \quad \text{and} \quad \liminf_{t\to 0+}\sum_{n=1}^{\infty} a_n\left(\frac{\sin nt}{nt}\right)^p=-\infty.$$

PROOF. We shall choose a sequence $\{N_k\}$ such that $N_1=1$, $2N_{k-1}<N_k$, and $N_k/25=$an integer when $k=2,3,4,\cdots$, and define a series $\sum_{n=1}^{\infty} a_n$ such that

$$a_n=\begin{cases} w_k/N_k & N_k<n\leqq \dfrac{26}{25}N_k \\ -w_k/2N_k & \dfrac{6}{5}N_k<n\leqq \dfrac{31}{25}N_k \\ 0 & \text{elsewhere,} \end{cases}$$

where $w_k=W_{N_k}$, when $k=1,2,3,\cdots$. Then it is easily seen that $\sum_{n=1}^{\infty} a_n=+\infty$. Let us now put $t_\nu=2\pi/N_\nu$ and write

$$\begin{aligned} f_p(t_\nu)&=\sum_{n=1}^{\infty} a_n\left(\frac{\sin nt_\nu}{nt_\nu}\right)^p=\sum_{k=1}^{\infty}\sum_{n=N_k}^{N_{k+1}-1} a_n\left(\frac{\sin nt_\nu}{nt_\nu}\right)^p \\ &=\sum_{k=1}^{\nu-1} U_k+U_\nu+\sum_{k=\nu+1}^{\infty} U_k=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3, \end{aligned} \tag{1}$$

say, where

$$U_k=\sum_{n=N_k}^{N_{k+1}-1} a_n\left(\frac{\sin nt_\nu}{nt_\nu}\right)^p.$$

For $\sum_1$, we have

$$\begin{aligned} |\textstyle\sum_1|&=|\sum_{k<\nu} U_k| \\ &\leqq \sum_{k<\nu}\left(\frac{w_k}{N_k}\sum_{n=N_k+1}^{\frac{26}{25}N_k}\left|\frac{\sin nt_\nu}{nt_\nu}\right|^p+\frac{1}{2}\frac{w_k}{N_k}\sum_{n=\frac{6}{5}N_k+1}^{\frac{31}{25}N_k}\left|\frac{\sin nt_\nu}{nt_\nu}\right|^p\right) \\ &\leqq \sum_{k<\nu}\frac{w_k}{N_k}\cdot\frac{1}{25}N_k+\frac{1}{2}\sum_{k<\nu}\frac{w_k}{N_k}\cdot\frac{1}{25}N_k \\ &=\frac{1}{25}\sum_{k<\nu} w_k+\frac{1}{50}\sum_{k<\nu} w_k<\frac{2}{25}\nu w_{\nu-1}. \end{aligned} \tag{2}$$

Now we estimate $\sum_2$.

$$\begin{aligned} \textstyle\sum_2&=\frac{w_\nu}{N_\nu}\sum_{n=N_\nu+1}^{\frac{26}{25}N_\nu}\left(\frac{\sin nt_\nu}{nt_\nu}\right)^p-\frac{w_\nu}{2N_\nu}\sum_{n=\frac{6}{5}N_\nu+1}^{\frac{31}{25}N_\nu}\left(\frac{\sin nt_\nu}{nt_\nu}\right)^p \\ &=\textstyle\sum_{21}-\sum_{22}, \text{ say.} \end{aligned}$$

In $\sum_{21}$, since $N_\nu<n\leqq \frac{26}{25}N_\nu$,

$$\left(\frac{\sin nt_\nu}{nt_\nu}\right)^p\leqq \frac{1}{(2\pi)^p}\sin^p\frac{2}{25}\pi\leqq\frac{1}{25^p},$$

and in $\sum_{22}$, since $\frac{30}{25}N_\nu<n\leqq\frac{31}{25}N_\nu$,

$$\left(\frac{\sin nt_\nu}{nt_\nu}\right)^p=\left(\frac{\sin(nt_\nu-2\pi)}{nt_\nu}\right)^p\geqq\left(\frac{25}{62\pi}\right)^p\sin^p\frac{10}{25}\pi\geqq\left(\frac{10}{31\pi}\right)^p.$$

Hence

$$\begin{aligned} \textstyle\sum_2&\leqq\left(\frac{1}{25}\right)^p\frac{w_\nu}{N_\nu}\cdot\frac{1}{25}N_\nu-\frac{w_\nu}{2N_\nu}\left(\frac{10}{31\pi}\right)^p\cdot\frac{1}{25}N_\nu \\ &=\frac{1}{25}\left\{\left(\frac{1}{25}\right)^p-\frac{1}{2}\cdot\left(\frac{10}{31\pi}\right)^p\right\}w_\nu \\ &\leqq -Aw_\nu, \end{aligned} \tag{3}$$

where A is a positive constant. Finally

$$
\begin{aligned}
|\sum_3| &= \left|\sum_{k>\nu}\sum_{n=N_k}^{N_{k+1}-1} a_n\left(\frac{\sin nt_\nu}{nt_\nu}\right)^p\right| \\
&\leqq \sum_{k>\nu}\frac{w_k}{N_k}\sum_{n=N_k+1}^{\frac{26}{25}N_k}\left|\frac{\sin nt_\nu}{nt_\nu}\right|^p + \sum_{k>\nu}\frac{w_k}{2N_k}\sum_{n=\frac{30}{25}N_k+1}^{\frac{31}{25}N_k}\left|\frac{\sin nt_\nu}{nt_\nu}\right|^p \\
&\leqq \sum_{k>\nu}\frac{w_k}{N_k^{p+1}t_\nu^p}\cdot\frac{1}{25}N_k + \sum_{k>\nu}\frac{w_k}{2N_k^{p+1}t_\nu^p}\cdot\frac{1}{25}N_k \\
&\leqq \sum_{k>\nu}\frac{w_k}{N_k^p t_\nu^p} = t_\nu^{-p}\sum_{k>\nu}\frac{w_k}{N_k^p} = t_\nu^{-p}\sum_{k>\nu}v_k,
\end{aligned}
$$

where $v_k = w_k/N_k^p$. Putting $\rho_k = v_k/v_{k-1}$, we have

$$
\begin{aligned}
|\sum_3| &\leqq t_\nu^{-p}v_\nu\sum_{k>\nu}\frac{v_k}{v_\nu} \\
&= t_\nu^{-p}v_\nu(\rho_{\nu+1}+\rho_{\nu+1}\,\rho_{\nu+2}+\rho_{\nu+1}\,\rho_{\nu+2}\,\rho_{\nu+3}+\cdots) \qquad (4)\\
&= o(w_\nu t_\nu^{-p}N_\nu^{-p}) = o(w_\nu),
\end{aligned}
$$

provided that

$$\rho_k \to 0 \quad \text{as} \quad k\to\infty.$$

Since

$$W_n n^{-p}\to 0 \quad\text{and}\quad W_n \nearrow \infty \quad \text{as} \quad n\to\infty,$$

we may choose the sequence $\{N_k\}$ such that

$$k w_{k-1}/w_k \to 0 \quad\text{and}\quad \rho_k\to 0 \quad\text{as}\quad k\to\infty.$$

Thus, by (1), (2), (3), and (4), we have

$$f_p(t_\nu) = o(w_\nu) - A w_\nu.$$

Therefore we get

$$\lim_{\nu\to\infty} f_p(t_\nu) = -\infty,$$

and then

$$\liminf_{t\to 0+} f_p(t) = -\infty,$$

which is the required result.

§ 3. Next we shall state the following theorem without the proof, since the proof is exactly similar to that of S.C. Lee's theorem. ([4, Theorem 1]).

THEOREM 2. Let $p\geqq 1$. Suppose that

$a_n \geqq -K/n$ ($n=1,2,3,\cdots$; K; a positive constant),

$\sum_{n=1}^{\infty} a_n = +\infty$ and $\sum_{n=1}^{\infty} a_n\left(\frac{\sin nt}{n}\right)^p$ converges in $0<t<t_0$.

Then

$$\lim_{t\to 0}\sum_{n=1}^{\infty} a_n\left(\frac{\sin nt}{nt}\right)^p = +\infty.$$

§ 4. Concerning the methods (R_p), we have the following theorems.

THEOREM 3. The method (R_{2p}) is totally regular.

PROOF. The proof is obvious if we use the I. Schur's theorem [1, p. 74]. But we give here a direct proof of the theorem. For the proof, it is sufficient to prove that if $\sum_{n=1}^{\infty} a_n = +\infty$, then

$$F_{2p}(t) = C_p^{-1} t \sum_{n=1}^{\infty} s_n \left(\frac{\sin nt}{nt}\right)^{2p} \longrightarrow \infty \quad \text{as} \quad t \longrightarrow 0+.$$

Since $\sum_{n=1}^{\infty} a_n = +\infty$, for an arbitrary positive number G, there exists an integer N_0 such that

$$s_n \equiv \sum_{k=1}^{n} a_k \geqq G \quad \text{when} \quad n \geqq N_0.$$

Now we take an arbitrary sequence $\{t_\nu\}$ such that $t_\nu \searrow 0$ as $\nu \longrightarrow \infty$. Let N_ν be the greatest integer less than or equal to π/t_ν. Then we may suppose $N_\nu > N_0$ for sufficiently large ν. Then we have

$$\begin{aligned} F_{2p}(t_\nu) &= C_p^{-1} t_\nu \sum_{n=1}^{\infty} s_n \left(\frac{\sin nt_\nu}{nt_\nu}\right)^{2p} \geqq C_p^{-1} t_\nu \sum_{n=1}^{N_\nu} s_n \left(\frac{\sin nt_\nu}{nt_\nu}\right)^{2p} \\ &\geqq C_p^{-1} G t_\nu \sum_{n=1}^{N_\nu} \left(\frac{\sin nt_\nu}{nt_\nu}\right)^{2p} - C_p^{-1} G t_\nu \sum_{n=1}^{N_0} \left(\frac{\sin nt_\nu}{nt_\nu}\right)^{2p} \\ &\quad + C_p^{-1} t_\nu \sum_{n=1}^{N_0} s_n \left(\frac{\sin nt_\nu}{nt_\nu}\right)^{2p} \\ &\longrightarrow C_p^{-1} G \int_0^{\pi} \left(\frac{\sin x}{x}\right)^{2p} dx \quad \text{as} \quad \nu \longrightarrow \infty. \end{aligned}$$

Since, G is arbitrary, we have

$$\lim_{t \to 0} F_{2p}(t) = +\infty,$$

which is the required result.

THEOREM 4. The method (R_{2p+1}) is not totally regular.

PROOF. For the proof, using a theorem due to H. Hurwitz [3, Theorem 6], it is sufficient to prove that

$$\lim_{t \to 0+} t \sum_{n=1}^{\infty} \left(\left| \frac{\sin nt}{nt} \right|^{2p+1} - \left(\frac{\sin nt}{nt}\right)^{2p+1} \right) > 0.$$

But this is easily proved using the definition of the definite integrals. See, for example, [2, Proof of Lemma 1]. Thus we have Theorem 4.

§5. Let α be a real number such that $-1 \leqq \alpha < p-1$, and let s_p^α be the Cesàro sum, of order α, of a series $\sum_{n=0}^{\infty} a_n$ with $a_0 = 0$. If the series in

$$\sigma(p, \alpha, t) = C_{p,\alpha}^{-1} t^{1+\alpha} \sum_{n=1}^{\infty} s_n^\alpha \left(\frac{\sin nt}{nt}\right)^p,$$

where

$$C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty u^{\alpha-p} \sin^p u \, du, & -1 < \alpha < p-1, \\ \pi/2 & \alpha = -1, \end{cases}$$

converges in some interval $0 < t < t_0$ and $\sigma(p, \alpha, t) \longrightarrow s$ as $t \longrightarrow 0+$, then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by the Riemann-Cesàro method of order p with index α, or shortly, summable (R, p, α) to s. This method of summation was introduced in my paper [2]. The method (R, p, α) is regular when $p \geqq 2$ and $-1 \leqq \alpha < p-1$. Concerning the total regularity, we have the following theorem which is proved by an argument similar to the direct proof of Theorem 3 in §4.

THEOREM 5. *The method $(R, 2p, \alpha)$ is totally regular when $0 \leqq \alpha < 2p-1$.*

References

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