6. *Axiom Systems of B-algebra. III*

By Shôtarô TANAKA

(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1966)

In this paper, we shall give an algebraic formulation of the axiom system of propositional calculus given by Lukasiewicz and Tarski (see [1]), and prove that this axiom system is equivalent to a B-algebra defined by K. Iséki (see [2].)

Let $\langle X, 0, *, \sim\rangle$ be an abstract algebra satisfying axioms:

(1) $x*w\leqslant(x*((((u*t)*(s*t)*((u*s)*r))*((\sim t*s)*\sim r)))*((y*z)*y)$.

(2) $0\leqslant x$.

D 1 If $x\leqslant y$ and $y\leqslant x$, then we put $x=y$.

D 2 $x\leqslant y$ means $x*y=0$.

(For details of the notions, see [2].)

In his paper [2], K. Iséki defines the notions of B-algebra $\langle X, 0, *, \sim\rangle$. The axioms are given by the following conditions:

B 1 $x*y\leqslant x$,

B 2 $(x*z)*(y*z)\leqslant(x*y)*z$,

B 3 $x*y\leqslant\sim y*\sim x$,

B 4 $0\leqslant x$,

and D 1, D 2.

Theorem. *A B-algebra is characterized by axioms* (1) *and* (2).

K. Iséki has proved that the axiom (1) is true in any B-algebra (see [3]). Therefore, we shall prove the converse. The fundamental ideas of the proof is due to my paper [4].

In axiom (1), we substitute z for w, $(x*y)*x$ for x and y, $(((u*t)*(s*t))*((u*s)*r))*((\sim t*s)*\sim r)$ for z, $((x*y)*x)*z$ appears in the left side. At the same time, the right side is equal to 0, because it is axiom (1) which is substituted $(((u*t)*(s*t))*((u*s)*r))*((\sim t*s)*\sim r)$ for w, $(x*y)*x$ for x, x for y and y for z in axiom (1) respectively. Therefore by (2), D 1 and D 2, we have

(3) $(x*y)*x\leqslant z$.

In this thesis, put $z=((x*y)*x)*z$, then by (2) and D 1, we have $(x*y)*x=0$. Hence by D 2, we have

(4) $x*y\leqslant x$.

Let us put $x=((((u*t)*(s*t))*((u*s)*r))*((\sim t*s)*\sim r))*((x*y)*x)$, $y=x$, $z=y$, $w=(x*y)*x$ in axiom (1), then the right side is equal to 0, because it is identical with the expression which is substituted $(((u*t)*(s*t))*((u*s)*r))*((\sim t*s)*\sim r)$ for x, $(x*y)*x$ for y, $(x*y)*x$ for z in (3). The second and third terms of the left side are equal to 0 by thesis (4). Therefore we have

(5) $((u*t)*(s*t))*((u*s)*r)\leqslant(\sim t*s)*\sim r$.

Putting $r=z$, $s=y$, $t=z$, and $u=x$ in (5), then we have, as the right side, $(z*y)*z$ which is equal to 0 by (4). Hence we have

(6) $(x*z)*(y*z)\leqslant(x*y)*z$.

In (6), put $x*y=0$, $y*z=0$, then by (2) we have $x*z=0$. Hence we have

(7) $x*y=0, y*z=0$, imply $x*z=0$, i.e., if $x\leqslant y$, $y\leqslant z$, then $x\leqslant z$. This means $(x*z)*(y*z)\leqslant x*y$.

Next put $z=x$, $y=z*y$ in (6), then we have

$$(x*x)*((z*y)*z)\leqslant(x*(z*y))*x.$$

The right side is 0 by (4), and the second term of the left side is 0 by (4), hence we have

(8) $x*x=0$, i.e., $x\leqslant x$.

If we put $x=x*z$, $y=y*z$, $z=x*y$ in (6), then we have

$$((x*z)*(x*y))*((y*z)*(x*y))\leqslant((x*z)*(y*z))*(x*y).$$

By (7), $((x*z)*(y*z))*(x*y)=0$, hence we have by (4)

$$((x*z)*(x*y))\leqslant(y*z)*(x*y)\leqslant y*z.$$

Therefore by (7), we have

(9) $((x*z)*(x*y))\leqslant y*z$, i.e., $y*z=0$ implies $x*z\leqslant x*y$.

In (4), if we put $x=(y*x)*y$, $y=(z*x)*(y*x)$, then we have

(10) $((y*x)*y)*((z*x)*(y*x))=0$.

In (6), let $x=(z*x)*y$, $y=(y*x)*y$, $z=(z*x)*(y*x)$, then by (7) and (10) we have

(11) $(z*x)*y\leqslant(z*x)*(y*x)$.

Next we shall prove a commutative law:

(12) $(z*x)*y=(z*y)*x$.

In (7), if we put $x=(z*y)*x$, $y=(z*y)*(x*y)$, $z=(z*x)*y$, then we have

$$\begin{aligned}&(((z*y)*x)*((z*x)*y))*(((z*y)*(x*y))*((z*x)*y))\\&\qquad\leqslant((z*y)*x))*((z*y)*(x*y)).\end{aligned}$$

The right side is 0 by (11) and the second term of the left side is 0 by (6). Hence we have

$$(z*y)*x\leqslant(z*x)*y.$$

In the above formula, if we put $x=y$, $y=x$, then we have

$$(z*x)*y\leqslant(z*y)*x.$$

Therefore by $D\,1$ we have a commutative law.

In the commutative law, let $x=(\sim y*z)*\sim x$, $y=(x*z)*x$, $z=(x*y)*(z*y)$, then we have by (5) and (4)

(13) $(x*y)*(z*y)\leqslant(\sim y*z)*\sim x$.

Putting $x=(x*y)*(z*y)$, $y=(\sim y*z)*\sim x$, $z=(\sim y*\sim x)*z$ in (9), and applying (12) to it, we have

(14) $(x*y)*(z*y)\leqslant(\sim y*\sim x)*z$.

In (12), if we put $x=(\sim y*\sim x)*y$, $y=y*y$, $z=x*y$ and apply (14), then we have

(15) $(x*y)\leqslant(\sim y*\sim x)*y$.

In (9), if we put $x=z$, $y=y*x$, $z=y$, then we have

$$((z*y)*(z*(y*x))\leqslant(y*x)*y=0.$$

Hence we have

(16) $z*y\leqslant z*(y*x)$.

In the above thesis, if we put $x=y$, $y=\sim y*\sim x$, $z=x*y$, then we have

$$(x*y)*(\sim y*\sim x)\leqslant(x*y)*((\sim y*\sim x)*y).$$

The right side is equal to 0 by (15). Hence we have

(17) $x*y\leqslant\sim y*\sim x$.

Theses (4), (6), and (17) hold in this news axiom sytem. Hence this new algebra is a B-algebra. The proof if complete. It is seen that this algebra is completely characterized by the expressions (4) and (5).

References

[1] J. Lukasiewicz und A. Tarski: Untersuchungen über den Aussagenkalkül. C. R. de Varsovie, C 1. III, **23**, 30-50 (1930).
[2] K. Iséki: Algebraic formulations of propositional calculi. Proc. Japan Acad., **41**, 803-807 (1965).
[3] ——: Some theorems in B-algebra. Proc. Japan Acad., **42**, 30-32 (1966).
[4] S. Tanaka: On axiom systems of propositional calculi. XIII. Proc. Japan Acad., **41**, 904-907 (1965).