45. Holomorphic Imbeddings of Symmetric Domains into a Symmetric Domain

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The problem of imbedding of symmetric domains into a symmetric domain is of interest in connection with the theories of moduli and of automorphic functions. Recently, Satake has determined all holomorphic imbeddings into a Siegel space ([3], [4]). The purpose of the present note is to treat the problem by a method similar to [3], and to determine in particular all holomorphic imbeddings into the exceptional domains (*EIII*) and (*EVII*). A more detailed paper will be published elsewhere.

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1. *Definitions and Notations* (following generally to those used in [3], [4]). A semi-simple Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ is called of *hermitian type* if a maximal compact subalgebra of each non-compact simple factor has non-trivial center. Let $G=\mathrm{Int}(\mathfrak{g})$ be the group of all inner automorphisms of $\mathfrak{g}$, K a subgroup of G corresponding to a maximal compact subalgebra $\mathfrak{k}$; let further $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition (corresponding to $\mathfrak{k}$). Then the symmetric space $D=G/K$ has a G-invariant complex structure and thus becomes a symmetric domain, and for each of such complex structures, there exists a uniquely determined H_0 in the center of $\mathfrak{k}$ such that $ad(H_0)$ induces on $\mathfrak{p}$, as the tangent space to D at the origin K, the given complex structure. Such an element H_0 is called an *H-element* of $\mathfrak{g}$ (relative to the Cartan decomposition). If $\mathfrak{g}$ is simple, D is irreducible and H-element is uniquely determined up to the sign $\pm$. The usual symbols $(I)_{p,q}$, $(II)_p$, $(III)_p$, $(IV)_p$, (*EIII*), and (*EVII*) for irreducible symmetric domain will be also used to denote the corresponding Lie algebras. By $\mathfrak{g}_C, \cdots$, we express the complexifications of $\mathfrak{g}, \cdots$.

All the fundamental properties of symmetric domains used in this paper will be found in [2].

2. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$ be semi-simple Lie algebras of hermitian type, and H_0, H_0' be H-elements of $\mathfrak{g}, \mathfrak{g}'$ respectively. We consider the problem in the following form (see [3]): *For given semi-simple Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *of hermitian type, determine all equivalence-classes of homomorphisms* ρ *of* $\mathfrak{g}$ *into* $\mathfrak{g}'$ *satisfying*

the condition

(H_1) $$\rho\circ ad(H_0)=ad(H_0')\circ\rho.$$

Two homomorphisms ρ_1 and ρ_2 of $\mathfrak{g}$ into $\mathfrak{g}'$ are equivalent if there is an element $s\in \mathrm{Int}(\mathfrak{g}')$ such that $\rho_2=s\circ\rho_1$; in particular, if we can take s in K', ρ_1 and ρ_2 are (k)-equivalent. The following proposition is a generalization of Corollary to Theorem 1 in [3].

Proposition. *Let ρ_1 and ρ_2 be homomorphisms of $\mathfrak{g}$ into $\mathfrak{g}'$ satisfying (H_1), where $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$ are semi-simple Lie algebras of hermitian type. Then ρ_1 and ρ_2 are equivalent if and only if they are (k)-equivalent.*

This is derived from the following

Lemma. *Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a semi-simple Lie algebra over $\boldsymbol{R}$ with a fixed Cartan decomposition, $\mathfrak{g}_1=\mathfrak{k}_1+\mathfrak{p}_1$ and $\mathfrak{g}_2=\mathfrak{k}_2+\mathfrak{p}_2$ semi-simple subalgebras of $\mathfrak{g}$ such that $\mathfrak{k}_i\subset\mathfrak{k}$ and $\mathfrak{p}_i\subset\mathfrak{p}$ $(i=1,2)$. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are conjugate, there is an element k in the subgroup of* Int$(\mathfrak{g})$ *corresponding to $\mathfrak{k}$ such that $\mathfrak{g}_2=k(\mathfrak{g}_1)$.*

3. If $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is of hermitian type, there is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$ and all such Cartan subalgebras are mutually conjugate by $K\subset \mathrm{Int}(\mathfrak{g})$. Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{k}$ once and for all, and denote by $\mathfrak{r}$ the root system of $\mathfrak{g}_\sigma$ relative to $\mathfrak{h}_\sigma$. Let H_0 be the H-element (fixed once for all). We shall always define an order of $\mathfrak{r}$ such that $\alpha(H_0)=1$ for a positive non-compact root. A subset Δ of $\mathfrak{r}$ is a *Π-system* if (i) $\alpha,\beta\in\Delta$ imply $\alpha-\beta\notin\Delta$, (ii) Δ is a linearly independent system. If Δ is a Π-system, we have a semi-simple subalgebra $\mathfrak{g}_\sigma(\Delta)$ of $\mathfrak{g}_\sigma$ called *regular subalgebra* ([1]). A Π-system Δ shall be called an *H-system* if a connected component (in the usual sense) contains no non-compact root or only one positive non-compact root.

Theorem 1. i) *Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a semi-simple subalgebra of hermitian type, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ in $\mathfrak{k}$, $\mathfrak{r}$ the root system of $\mathfrak{g}_\sigma$ relative to $\mathfrak{h}_\sigma$, and suppose that an H-system Δ in $\mathfrak{r}$ is given. Then the semi-simple subalgebra $\mathfrak{g}_\sigma(\Delta)$ is defined over $\boldsymbol{R}$, $\mathfrak{g}(\Delta)=\mathfrak{g}_\sigma(\Delta)\cap\mathfrak{g}$ is of hermitian type, and there is an H-element of $\mathfrak{g}(\Delta)$ such that the injection homomorphism $\iota\colon\mathfrak{g}(\Delta)\to\mathfrak{g}$ satisfies (H_1).*

ii) *If Δ_1 and Δ_2 are H-systems in $\mathfrak{r}$, $\mathfrak{g}(\Delta_1)$ and $\mathfrak{g}(\Delta_2)$ are conjugate (in $\mathfrak{g}$), if and only if there is an element w in the Weyl group of $\mathfrak{k}$ (as a subgroup of the Weyl group of $\mathfrak{g}_\sigma$) such that $\Delta_2=w(\Delta_1)$.*

If Δ is an H-system, the semi-simple subalgebra $\mathfrak{g}(\Delta)=\mathfrak{g}_\sigma(\Delta)\cap\mathfrak{g}$ of $\mathfrak{g}$ will be called an *H-subalgebra*.

4. Let the notations be the same as in 2. The following (H_2) is a stronger condition than (H_1);

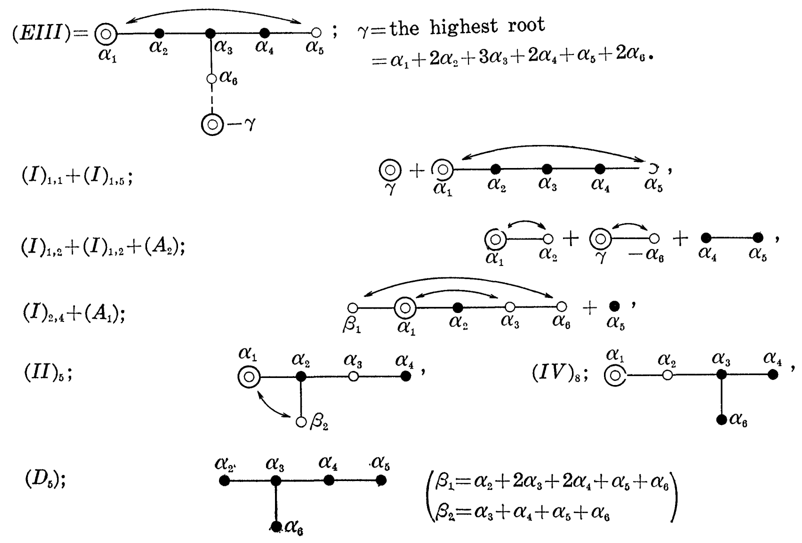
(H_2) $$\rho(H_0)=H_0'.$$

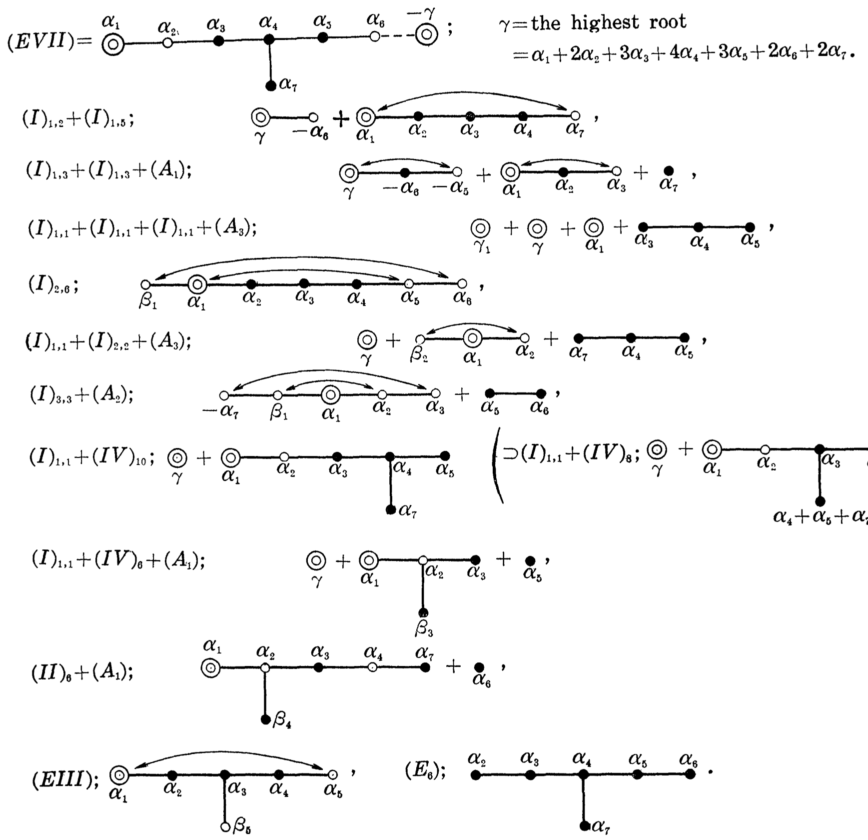
The following Theorem 2 is essentially proved in the Proposition 1 of [3].

Theorem 2. *Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be semi-simple Lie algebras of hermitian types,* ρ *a homomorphism of* $\mathfrak{g}$ *into* $\mathfrak{g}'$ *which does not satisfy* (H_2) *but* (H_1). *Then we have an H-subalgebra* $\mathfrak{g}''$ *of* $\mathfrak{g}'$ *such that the image of* ρ *is contained in* $\mathfrak{g}''$, *and* $\rho: \mathfrak{g}\to\mathfrak{g}''$ *satisfies* (H_2).

Our problem is easily reduced to the case where $\mathfrak{g}'$ is simple. Hence it is sufficient for our purpose to find all conjugate classes of H-subalgebras of simple $\mathfrak{g}'$ and to determine all semi-simple Lie algebras $\mathfrak{g}$ of hermitian type and all homomorphisms $\rho: \mathfrak{g}\to\mathfrak{g}'$ satisfying (H_2). If $\mathfrak{g}'$ is classical, all pairs $(\mathfrak{g}, \rho)$ satisfying (H_2) are already determined by Satake ([3], [4]), and it will be easy to find by straightforward calculations all equivalence classes (under the Weyl group of $\mathfrak{k}'$) of H-systems of $\mathfrak{g}'$.

5. Now we give here the complete solution for the case $\mathfrak{g}'=(EIII)$ or $(EVII)$. In these cases, all the positive non-compact roots are mutually permutable by translations defined by the Weyl group of $\mathfrak{k}'$. Hence one of the positive non-compact roots in an H-system may be assumed to be the non-compact simple root of $\mathfrak{g}'$. Moreover, since the rank of the symmetric domain $(EIII)$ (resp. $(EVII)$) is 2 (resp. 3), the number of the positive non-compact roots in an H-system is at most 2 (resp. 3); in other words, the number of non-compact factors of an H-subalgebra of $(EIII)$ (resp. $(EVII)$) is at most 2 (resp. 3). All classes of the *maximal* H-system of $(EIII)$ and $(EVII)$ are as follows (the diagrams are Dynkin-Satake diagrams of Lie algebras over $\boldsymbol{R}$, and double circles in them show non-compact

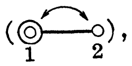


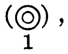


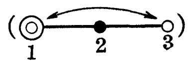
$$\left(\begin{array}{ll} \beta_1=\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_7, & \beta_2=\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7 \\ \beta_3=\alpha_3+2\alpha_4+\alpha_5+\alpha_7, & \beta_4=\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7 \\ \beta_5=\alpha_4+\alpha_5+\alpha_6+\alpha_7, & \gamma_1=\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_7 \end{array}\right)$$

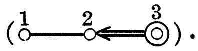
roots): All the other H-subalgebras are contained in some of them.

Let ι be the injection of an H-subalgebra $\mathfrak{g}''$ into $(EIII)$ (resp. $(EVII)$). If $\mathfrak{g}''$ is $(I)_{1,1}+(I)_{1,5}$, $(I)_{1,2}+(I)_{1,2}$, or $(I)_{2,4}$ (resp. $(I)_{1,2}+(I)_{1,5}$, $(I)_{1,3}+(I)_{1,3}$, $(I)_{1,1}+(I)_{1,1}+(I)_{1,1}$, $(I)_{2,6}$, $(I)_{3,3}$, or $(I)_{1,1}+(IV)_{2p}$; $3\leq p\leq 5$) up to compact factors, ι satisfies (H_2). In addition to them, there are some pairs (g, ρ) which satisfy (H_2). By the fundamental representation φ of complex Lie algebra (E_6) (resp. (E_7)) associated to α_1, we can realize the Lie algebra $(EIII)$ (resp. $(EVII)$) in $\mathfrak{gl}\,(27, \boldsymbol{C})$ (resp. $\mathfrak{gl}\,(56, \boldsymbol{C})$). Writing now ρ instead of $\varphi\circ\rho$, we have a representation of $(EIII)$ (resp. $(EVII)$) of dimension 27 (resp. 56). We find the following pairs which satisfy (H_2): (λ_i: the highest weight of a fundamental representation).

$(EIII)$: $\mathfrak{g}=(I)_{1,2}$, $\rho=(2\lambda_1)\dotplus 7(\lambda_2)$ 

$(EVII)$: $\mathfrak{g}=(I)_{1,1}$, $\rho=(3\lambda_1)\dotplus 26(\lambda_1)$ 

$\mathfrak{g}=(I)_{1,3}$, $\rho=(2\lambda_1)\dotplus(2\lambda_3)\dotplus 6(\lambda_2)$ 

$\mathfrak{g}=(III)_3$, $\rho=7(\lambda_1)\dotplus(\lambda_3)$ 

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