106. A Generalization of the Cauchy Filter and the Completion

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In this paper, to take away the notion of covering system and we consider about the completion theory of topological space with a set consisting of some filters instead of Cauchy filters concerning covering system.

Thus, we get a generalization of author's paper [5], but using method is not different almost at all.

By this generalization, Alexandroff one point compactification is included, as a special case, in the completion.

A family $\mathfrak{f}$ consisting of subsets of X is a *filter base* in X if for every $A, B \in \mathfrak{f}, C \subseteq A \cap B$ for some $C \in \mathfrak{f}$ and $\phi \notin \mathfrak{f}$.

A *filter* $\mathfrak{f}$ in X is a filter base in X such that if $A \supseteq B$ and $B \in \mathfrak{f}$ then $A \in \mathfrak{f}$.

For every filter base $\mathfrak{f}$ in X, the family $\{A \mid X \supseteq A \supseteq B, B \in \mathfrak{f}\}$ is a filter in X, that is said to be *generated* by $\mathfrak{f}$.

If $X^* \supseteq X$ then a filter $\mathfrak{f}$ in X is a filter base in X^* and generates a filter in X^*. Denote it by $\mathfrak{f}^*$.

In a topological space X, let's denote by $\mathfrak{N}(x)$, the neighborhood system of $x \in X$, and by $\mathfrak{G}(X)$, the family of all open sets of X.

A filter base $\mathfrak{f}$ in a topological space X *converges* to x in X if and only if the filter generated by $\mathfrak{f}$ contains the neighborhood system $\mathfrak{N}(x)$ of x.

For a filter base $\mathfrak{f}$ in a topological space X, $\{G \mid G \in \mathfrak{G}(X), G \supseteq A, A \in \mathfrak{f}\}$ is a filter base, so generates a filter, we will denote it by $\mathfrak{f}^\pi$. Thus $\mathfrak{f}^\pi$ converges to x if and only if $\mathfrak{f}$ converges to x.

We consider a topological space X, with a set M consisting of some filters that satisfies the following three conditions

M1) if $\mathfrak{f} \in M$ and $\mathfrak{g} \supseteq \mathfrak{f}$ then $\mathfrak{g} \in M$,

M2) if $\mathfrak{f} \in M$ then $\mathfrak{f}^\pi \in M$,

M3) for all point x of X, $\mathfrak{N}(x) \in M$.

Let's denote such a topological space X, by $(X; M)$. In $(X; M)$, if $\mathfrak{f} \in M$ converges to no point, then $\mathfrak{f}$ is a *leg*. If $(X; M)$ has no leg, $(X; M)$ is *complete*.

A *completion* $(X^*; M^*)$ of a space $(X; M)$ is such a space that

C1) $X \subseteq X^*$,

C2) $(X^*; M^*)$ is complete,
C3) for every open set G of X, there corresponds an open set G^* of X^* such that $G^* \cap X = G$, and $\{G^* \mid G \in \mathfrak{G}(X)\}$ become a base of open sets of X^*,
C4) for every $\mathfrak{f} \in M^*$, $\{A \cap X \mid A \in \mathfrak{f}^\pi\} \in M$, and for every $\mathfrak{f} \in M$, $\mathfrak{f}^* \in M^*$,
C5) if for every $\mathfrak{f} \in M$ converging to a point x of X^*, $G \in \mathfrak{f} \cap \mathfrak{G}(X)$, then $x \in G^*$,
C6) $\mathfrak{f} \in M$ converges to only one point in X^* and for every point x of $X^* \sim X$, there exists at least one leg converging to x.

Put $\mathfrak{G}^* = \{G^* \mid G \in \mathfrak{G}(X)\}$.

Let there exists a completion $(X^*; M^*)$ of $(X; M)$.

If a leg $\mathfrak{f} \in M$ containing a leg $\mathfrak{g} \in M$ converges to x in X^*, then $\mathfrak{g}$ also converges to x in X^*, by the condition C6). So the filter $\bigcap_{\mathfrak{g} \subseteq \mathfrak{f}, \mathfrak{g} \in M} \mathfrak{g}$ converges to x too. Let's denote $\bigcap_{\mathfrak{g} \subseteq \mathfrak{f}, \mathfrak{g} \in M} \mathfrak{g}$ by $[\mathfrak{f}]$. Thus, using M3), we obtain $[\mathfrak{f}]^* \in M^*$ and moreover by C4), $\{A \cap X \mid A \in [\mathfrak{f}]^{*\pi}\} \in M$. From the fact $[\mathfrak{f}] \supseteq \{A \cap X \mid A \in [\mathfrak{f}]^{*\pi}\}$ and M1), we get $[\mathfrak{f}] \in M$.

Thus, the following proposition holds:
E) if $\mathfrak{f}$ is a leg then $[\mathfrak{f}]$ is a leg too.

M2) shows that for all leg $\mathfrak{f}$, $[\mathfrak{f}]^\pi = [\mathfrak{f}]$.

A member of $[\mathfrak{f}] \cap \mathfrak{G}(X)$ is called a *body* of $\mathfrak{f}$.

Then the condition C5) is equivalent to; $G \in \mathfrak{G}(X)$ is a body of a leg $\mathfrak{f} \in M$ if and only if for the point x to which $\mathfrak{f}$ converges in X^*, $x \in G^* \sim G$.

Now, assume a space $(X; M)$ satisfies the above condition E).

A leg $\mathfrak{f}$ is *minimal* if and only if $\mathfrak{f} = [\mathfrak{f}]$.

For every $G \in \mathfrak{G}(X)$, denote by $\varphi(G)$, the set $\{\mathfrak{f} \mid \mathfrak{f}$; minimal leg, $G \in \mathfrak{f}\}$. Thus for every open sets G, H of X, we get $\varphi(G) \cap \varphi(H) = \varphi(G \cap H)$. Put $\varphi(G) \cup G = G^*$. So $\phi^* = \phi$ and $G^* \cap H^* = (G \cap H)^*$. These show that $\mathfrak{G}^* = \{G^* \mid G \in \mathfrak{G}(X)\}$ is a base of open sets of X^*.

We define the set M^* as $M^* = \{\mathfrak{f} \mid \mathfrak{f}$; filter in X^*, $\{A \cap X \mid A \in \mathfrak{f}^\pi\} \in M\}$. It is easy to see this M^* satisfies the conditions M1), M2), M3), and C4).

For $\mathfrak{f}' \in M^*$, put $\mathfrak{f} = \{A \cap X \mid A \in \mathfrak{f}'^\pi\}$. Then $\mathfrak{f} \in M$ and easily seeing, an open set G of X belongs to $\mathfrak{f}$ if and only if G^* belongs to $\mathfrak{f}'$. Either $\mathfrak{f}$ converges to a point x in X or $\mathfrak{f}$ is a leg in X. If $\mathfrak{f}$ converges to a point x in X then $\mathfrak{f}'$ converges to x in X^*, on the other hand if $\mathfrak{f}$ is a leg in X then $\mathfrak{f}'$ converges to $[\mathfrak{f}]$ in X^*. This shows that $(X^*; M^*)$ is complete.

If $\mathfrak{f}$ converges to a minimal leg $\mathfrak{g}$ in X^*, then $[\mathfrak{f}] = \mathfrak{g}$ and furthermore $\mathfrak{f}$ never converges to any point of X in X^*. So

$(X^*; M^*)$ satisfies the former part of the condition C6). Other conditions are satisfied from our construction of the space $(X^*; M^*)$. So we get

Theorem 1. *$(X; M)$ has its completion if and only if the following is satisfied; if $\mathfrak{f}$ is a leg then $[\mathfrak{f}]$ is also a leg.*

Let $(X^*; M^*)$ and $(X^+; M^+)$ are both completions of $(X; M)$. Then there exists a mapping f of $(X^*; M^*)$ onto $(X^+; M^+)$ such that $f(x)=x$ for every $x \in X$ and for any $x \in X^* \sim X$, $\{A \cap X \mid A \in \mathfrak{N}(x)\}$ is a leg in X and converges to some point y in X^+, thus $f(x)=y$.

Then, for every $G \in \mathfrak{G}(X)$, $f(G^*)=G^+$ by the condition C5), which shows that f is topological and so we obtain;

Theorem 2. *A completion is uniquely determined by a space $(X; M)$.*

Let $(X^*; M^*)$ be a completion of $(X; M)$ and f be a continuous mapping of $(X; M)$ into a topological space Y such that for every $y \in Y$, and for any neighborhood V of y, there exists some neighborhood U of y and $\bar{U} \subseteq V$. If for every leg $\mathfrak{f} \in M$, $\{f(A) \mid A \in \mathfrak{f}\}$ converges to some point of Y, then f is extendible on $(X^*; M^*)$; there exits a extention F of f, provided for every $x \in X^* \sim X$, $F(x)$ is arbitraly point in Y, to which $\{f(A) \mid A \in \mathfrak{f}\}$ converges, for the minimal leg $\mathfrak{f}$ converging to x in X^*.

Theorem 3. *Let f is a continuous mapping of $(X; M)$ to have completion $(X^*; M^*)$ into a topological space Y such that for every $y \in Y$ and for every $V \in \mathfrak{N}(y)$, $\bar{U} \subseteq V$ for some $U \in \mathfrak{N}(y)$. Then there exists a continuous mapping F satisfing that for every $x \in X$, $f(x)=F(x)$, if and only if $\{f(A) \mid A \in \mathfrak{f}\}$ converges to some point of Y, for every leg $\mathfrak{f} \in M$.*

Next, we consider the product space of our space $(X_\lambda; M_\lambda)$, $\lambda \in \Delta$.

The *product* $(X; M)$ of our space $(X_\lambda; M_\lambda)$ is such that;

P1) $X = \prod X_\lambda$ and X has the weak topology,

P2) $\mathfrak{f} \in M$ if and only if $\{P_\lambda(A) \mid A \in \mathfrak{f}\} \in M_\lambda$, provided, P_λ is the projection of X into its λ-component X_λ.

Let's denote the product of $(X_\lambda; M_\lambda)$ by $\prod(X_\lambda; M_\lambda)$. Above M satisfies obviously the conditions M1), M2), and M3).

As X has weak topology, $\mathfrak{f} \in M$ converges to $x \in X$ if and only if $\{P_\lambda(A) \mid A \in \mathfrak{f}\}$ converges to $P_\lambda(x) \in X_\lambda$. Thus we get; a product is complete if and only if its every component is complete.

Theorem 4. *A product $\prod(X_\lambda; M_\lambda)$ of $(X_\lambda; M_\lambda)$ is complete if and only if every component $(X_\lambda; M_\lambda)$ is complete.*

A filter $\mathfrak{f} \in M$ is *minimal* in M if and only if there are no filter of M that is properly contained in $\mathfrak{f}$.

Then, in a product $(X; M)$ of spaces $(X_\lambda; M_\lambda)$, $\mathfrak{f} \in M$ is minimal if and only if $\{P_\lambda(A) \mid A \in \mathfrak{f}\}$ is minimal in M_λ and $\mathfrak{f}$ is generated by $\{\prod P_\lambda(A) \mid A \in \mathfrak{f}\}$.

In T_2 space, every filter converges to at most one point.

These results show that;

Theorem 5. *Assume that $(X_\lambda^*; M_\lambda^*)$ is the completion of $(X_\lambda; M_\lambda)$ and they are both T_2. Then the product $\prod(X_\lambda^*; M_\lambda^*)$ of $(X_\lambda^*; M_\lambda^*)$ is the completion of the product $\prod(X_\lambda; M_\lambda)$ of $(X_\lambda; M_\lambda)$.*

In general, for every open sets G and H, if $G \cap H = \phi$ then $G^* \cap H^* = \phi$.

If there are two legs $\mathfrak{f}$ and $\mathfrak{g}$ such as for every bodies $V \in \mathfrak{f}$ and $W \in \mathfrak{g}$, $V \cap W \neq \phi$, then $\{V \cap W \mid V \in \mathfrak{f}, W \in \mathfrak{g}\}$ is also in M. Either this filter converges or is leg. If it is a leg then $[\mathfrak{f}] = [\{V \cap W \mid V \in \mathfrak{f}, W \in \mathfrak{g}\}] = [\mathfrak{g}]$. So we get

Proposition. *The completion $(X^*; M^*)$ of T_2 space $(X; M)$ is T_2 if and only if for any point $x \in X$ and for every leg $\mathfrak{f} \in M$, $V \cap W = \phi$ for some neighborhood $V \in \mathfrak{N}(x)$ and some body W of $\mathfrak{f}$.*

References

[1] N. Bourbaki: Topologie Generale, Paris (1940).
[2] L. Cohen: Uniformity property in topological space satisfing the first denumerability postulate. Duke Math. Jour., **3** (1937).
[3] ——: On imbedding space in a complete space. Duke Math. Jour., **5** (1939).
[4] T. Inagaki: Point Set Theory (Japanese). Tokyo (1949).
[5] S. Mitani: A note on the completion theory. Proc. Japan Acad., **39**, 270–273 (1963).
[6] E. Moore and H. Smith: A general theory of limits. Amer. Jour. Math., **44** (1922).
[7] J. Kelley: General Topology. New York (1955).
[8] J. Tukey: Convergence and Uniformity in Topology. Princeton (1940).