130. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XXI

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Definition A. Let $T(\lambda)$ be the function stated in [1]; let $\sigma=\sup_{\nu}|\lambda_\nu|$; and let the mutually disjoint, closed, and connected domains D_j $(j=1, 2, 3, \cdots, n)$ which have no point in common with the closure of the denumerably infinite set $\{\lambda_\nu\}_{\nu=1,2,3,\ldots}$ be contained in the disc $|\lambda|\leqq\sigma$. Hence, by definition, $T(\lambda)$ is regular in the complex λ-plane $\{\lambda: |\lambda|<+\infty\}$ with the exception of $\overline{\{\lambda_\nu\}}\cup\left[\bigcup_{j=1}^{n}D_j\right]$ and every point belonging to the set $\overline{\{\lambda_\nu\}}\cup\left[\bigcup_{j=1}^{n}D_j\right]$ is a singularity of $T(\lambda)$. Here $\overline{\{\lambda_\nu\}}$ denotes the closure of $\{\lambda_\nu\}$.

Theorem 59. Let
$$m(\rho, \infty)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|T(\rho e^{-it})|dt \quad (\sigma<\rho<+\infty).$$
Then
$$\varlimsup_{\rho\to\sigma+0}\frac{m(\rho, \infty)}{\log\dfrac{1}{\rho-\sigma}}<+\infty.$$

Proof. Since, as already stated in [1], the sum-function $\chi(\lambda)$ of the first and second principal parts of $T(\lambda)$ is given by
$$\begin{aligned}\chi(\lambda)&=\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}+f_{2\alpha}), (f'_{1\alpha}+f'_{2\alpha}))+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta})\\ &=\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}\frac{c_\alpha^{(\nu)}}{(\lambda-\lambda_\nu)^\alpha}+\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}f_{2\alpha}, f'_{2\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta})\\ &\qquad\qquad (1\leqq m, n, k_j<+\infty),\end{aligned}$$
where $\sum_{\nu=1}^{\infty}|c_\alpha^{(\nu)}|\leqq\|f_{1\alpha}\|\,\|f'_{1\alpha}\|<+\infty$, we can find from the inequality $\overset{+}{\log}\left|\sum_{\mu=1}^{p}\alpha_\mu\right|\leqq\sum_{\mu=1}^{p}\overset{+}{\log}|\alpha_\mu|+\log p$ holding for any complex numbers α_μ that
$$\begin{aligned}\overset{+}{\log}|T(\rho e^{-it})|&\leqq\overset{+}{\log}|R(\rho e^{-it})|\\ &+\overset{+}{\log}\left|\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}\frac{c_\alpha^{(\nu)}}{(\rho e^{-it}-\lambda_\nu)^\alpha}\right|+\overset{+}{\log}\left|\sum_{\alpha=1}^{m}((\rho e^{-it}I-N_1)^{-\alpha}f_{2\alpha}, f'_{2\alpha})\right|\\ &+\overset{+}{\log}\left|\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\rho e^{-it}I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta})\right|+\log 4 \quad (\sigma<\rho<+\infty),\end{aligned}$$
where $R(\lambda)$ denotes the ordinary part of $T(\lambda)$ and hence is an integral

function by definition. Since, on the other hand,

$$\overset{+}{\log}\left|\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}\frac{c_\alpha^{(\nu)}}{(\rho e^{-it}-\lambda_\nu)^\alpha}\right| \leqq \sum_{\alpha=1}^{m}\overset{+}{\log}\frac{\sum_{\nu=1}^{\infty}|c_\alpha^{(\nu)}|}{(\rho-\sigma)^\alpha}+\log m \quad (\sigma<\rho<+\infty)$$

$$\leqq \sum_{\alpha=1}^{m}\left\{\overset{+}{\log}\frac{1}{(\rho-\sigma)^\alpha}+\overset{+}{\log}\sum_{\nu=1}^{\infty}|c_\alpha^{(\nu)}|\right\}+\log m,$$

$$\text{(47)} \qquad \varlimsup_{\rho\to\sigma+0}\frac{\overset{+}{\log}\left|\sum_{\alpha=1}^{m}\sum_{\nu=1}^{\infty}\frac{c_\alpha^{(\nu)}}{(\rho e^{-it}-\lambda_\nu)^\alpha}\right|}{\log\frac{1}{\rho-\sigma}} \leqq \frac{m(m+1)}{2}.$$

Suppose now that $\{K^{(j)}(z)\}$ denotes the complex spectral family of the bounded normal operator N_j for each value of $j=1, 2, 3, \cdots, n$. Then there is no difficulty in showing that $\left\|\sum_{k=1}^{p}\varepsilon_k K^{(1)}(\Delta_k)f_{2\alpha}\right\|^2=\left\|\sum_{k=1}^{p}K^{(1)}(\Delta_k)f_{2\alpha}\right\|^2$ for any ε_k with modulus 1 and any mutually disjoint (bounded) domains Δ_k, $k=1, 2, 3, \cdots, p$; and hence we can verify from the definitions concerning $f_{2\alpha}$, $f'_{2\alpha}$, D_1, $\{\lambda_\nu\}$, and σ that

$$|((\rho e^{-it}I-N_1)^{-\alpha}f_{2\alpha}, f'_{2\alpha})|=\left|\int_{[D_1\cup\overline{\{\lambda_\nu\}}]-\{\lambda_\nu\}}\frac{1}{(\rho e^{-it}-z)^\alpha}d(K^{(1)}(z)f_{2\alpha}, f'_{2\alpha})\right|$$

$$\leqq \frac{1}{(\rho-\sigma)^\alpha}\,||\,K^{(1)}([D_1\cup\overline{\{\lambda_\nu\}}]-\{\lambda_\nu\})f_{2\alpha}\,||\,||\,f'_{2\alpha}\,||$$

$$\leqq \frac{||\,f_{2\alpha}\,||\,||\,f'_{2\alpha}\,||}{(\rho-\sigma)^\alpha}\,(\sigma<\rho<+\infty).$$

This last result yields the inequalities

$$\overset{+}{\log}\left|\sum_{\alpha=1}^{m}((\rho e^{-it}I-N_1)^{-\alpha}f_{2\alpha}, f'_{2\alpha})\right|$$

$$\leqq \sum_{\alpha=1}^{m}\overset{+}{\log}\frac{||\,f_{2\alpha}\,||\,||\,f'_{2\alpha}\,||}{(\rho-\sigma)^\alpha}+\log m \quad (\sigma<\rho<+\infty)$$

$$\leqq \sum_{\alpha=1}^{m}\left\{\overset{+}{\log}\frac{1}{(\rho-\sigma)^\alpha}+\overset{+}{\log}\,||\,f_{2\alpha}\,||\,||\,f'_{2\alpha}\,||\right\}+\log m,$$

so that

$$\text{(48)} \qquad \varlimsup_{\rho\to\sigma+0}\frac{\overset{+}{\log}\left|\sum_{\alpha=1}^{m}((\rho e^{-it}I-N_1)^{-\alpha}f_{2\alpha}, f'_{2\alpha})\right|}{\log\frac{1}{\rho-\sigma}} \leqq \frac{m(m+1)}{2}.$$

Since, moreover, it is similarly verified from the definitions of D_j, $g_{j\beta}$, and $g'_{j\beta}$ that

$$|((\rho e^{-it}I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta})|=\left|\int_{D_j}\frac{1}{(\rho e^{-it}-z)^\beta}d(K^{(j)}(z)g_{j\beta}, g'_{j\beta})\right| (\sigma<\rho<+\infty)$$

$$\leqq \frac{||\,g_{j\beta}\,||\,\,||\,g'_{j\beta}\,||}{(\rho-\sigma)^\beta},$$

we have

$$\overset{+}{\log}\left|\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\rho e^{-it}I-N_j)^{-\beta}g_{j\beta},\ g'_{j\beta})\right|$$
$$\leqq\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}\overset{+}{\log}\frac{\|g_{j\beta}\|\,\|g'_{j\beta}\|}{(\rho-\sigma)^{\beta}}+\log\sum_{j=2}^{n}k_j\quad(\sigma<\rho<+\infty)$$
$$\leqq\sum_{j=2}^{n}\frac{k_j(k_j+1)}{2}\cdot\overset{+}{\log}\frac{1}{\rho-\sigma}+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}\overset{+}{\log}\|g_{j\beta}\|\,\|g'_{j\beta}\|+\log\sum_{j=2}^{n}k_j$$

and hence

$$\overline{\lim_{\rho\to\sigma+0}}\frac{\overset{+}{\log}\left|\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\rho e^{-it}I-N_j)^{-\beta}g_{j\beta},\ g'_{j\beta}\right|}{\log\dfrac{1}{\rho-\sigma}}\leqq\sum_{j=2}^{n}\frac{k_j(k_j+1)}{2}. \tag{49}$$

Remembering that $R(\lambda)$ is an integral function, it is found from the inequalities (47), (48), and (49) that

$$\overline{\lim_{\rho\to\sigma+0}}\frac{m(\rho,\infty)}{\log\dfrac{1}{\rho-\sigma}}\leqq m(m+1)+\sum_{j=2}^{n}\frac{k_j(k_j+1)}{2}<+\infty,$$

as we wished to prove.

Definition B. From now on we shall suppose that m is not finite. In the first place, $f_{1\alpha}$ and $f'_{1\alpha}$ are expressed in the forms $f_{1\alpha}=\sum_{\nu=1}^{\infty}a_{1\alpha}^{(\nu)}\varphi_{\nu}^{(1)}\in\mathfrak{M}_1$ and $f'_{1\alpha}=\sum_{\nu=1}^{\infty}\tilde{a}_{1\alpha}^{(\nu)}\varphi_{\nu}^{(1)}\in\mathfrak{M}_1$ respectively [1]; and here $\sum_{\nu=1}^{\infty}|a_{1\nu}^{(\alpha)}|^2<+\infty$ and $\sum_{\nu=1}^{\infty}|\tilde{a}_{1\alpha}^{(\nu)}|^2<+\infty$. We next suppose that the multiplicity of any eigenvalue $\lambda_\nu\in\{\lambda_\nu\}$ of N_1 equals 1. Since, by definition, $N_1\varphi_{\nu}^{(1)}=\lambda_\nu\varphi_{\nu}^{(1)}(\nu=1,2,3,\cdots)$, it is clear that $c_{\alpha}^{(\nu)}=a_{1\alpha}^{(\nu)}\overline{\tilde{a}_{1\alpha}^{(\nu)}}$ for $c_{\alpha}^{(\nu)}$ in the equality $((\lambda I-N_1)^{-\alpha}f_{1\alpha},f'_{1\alpha})=\sum_{\nu=1}^{\infty}\dfrac{c_{\alpha}^{(\nu)}}{(\lambda-\lambda_\nu)^{\alpha}}$. In addition, $f_{1\alpha}$ and $f'_{1\alpha}$ can be so chosen as to satisfy the condition

$$\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{|a_{1\alpha}^{(\nu)}\overline{\tilde{a}_{1\alpha}^{(\nu)}}|}{|\lambda-\lambda_\nu|^{\alpha}}<+\infty\ (a_{1\alpha}^{(\nu)}\overline{\tilde{a}_{1\alpha}^{(\nu)}}\neq0,\ \alpha,\ \nu=1,2,3,\cdots;\ \lambda\notin\overline{\{\lambda_\nu\}}). \tag{50}$$

If, for example, we set $a_{1\alpha}^{(\nu)}=\dfrac{K_1}{\sqrt{\alpha!}\,c_1^{\nu}}$ and $\tilde{a}_{1\alpha}^{(\nu)}=\dfrac{K_2}{\sqrt{\alpha!}\,c_2^{\nu}}$ where $1<c_i<+\infty$ and $0<|K_i|<+\infty$ for $i=1,2$, then

$$\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}f_{1\alpha},f'_{1\alpha})=\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{K_1\bar{K}_2}{(c_1c_2)^{\nu}\cdot\alpha!(\lambda-\lambda_\nu)^{\alpha}}\ (\lambda\notin\overline{\{\lambda_\nu\}}).$$

Since, moreover, we have

$$\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{|K_1\bar{K}_2|}{(c_1c_2)^{\nu}\cdot\alpha!|\lambda-\lambda_\nu|^{\alpha}}\leqq\frac{|K_1\bar{K}_2|}{c_1c_2-1}(e^{M}-1)\ (\lambda\notin\overline{\{\lambda_\nu\}})$$

where $M=\sup_{\nu}\dfrac{1}{|\lambda-\lambda_\nu|}$, condition (50) is satisfied. Moreover $f_{1\alpha}$ and $f'_{1\alpha}$ both belong to $\mathfrak{M}_1$ for such $a_{1\alpha}^{(\nu)}$ and $\tilde{a}_{1\alpha}^{(\nu)}$ $(\nu=1,2,3,\cdots)$. On the other hand, we can show as below that for any pair of $f_{1\alpha}\in\mathfrak{M}_1$ and $f'_{1\alpha}\in\mathfrak{M}_1$ satisfying (50) the function $\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}f_{1\alpha},f'_{1\alpha})$ is regular in the domain $\mathfrak{D}\{\lambda:\lambda\notin\overline{\{\lambda_\nu\}}\}$. Since both $f_{1\alpha}$ and $f'_{1\alpha}$ belong to the subspace $\mathfrak{M}_1$ determined by the incomplete orthonormal set $\{\varphi_\nu^{(1)}\}$,

$$T_m(\lambda)\equiv\sum_{\alpha=1}^{m}((\lambda I-N_1)^{-\alpha}f_{1\alpha}, f'_{1\alpha}) \quad (\lambda\notin\overline{\{\lambda_\nu\}},\ 1\leqq m<+\infty)$$
$$=\sum_{\alpha=1}^{m}\int_{\{\lambda_\nu\}}\frac{1}{(\lambda-z)^\alpha}d(K^{(1)}(z)f_{1\alpha}, f'_{1\alpha})$$

and hence $T_m(\lambda)$ is regular in $\mathfrak{D}$. Furthermore it is easily verified that the infinite sequence $\{T_m(\lambda)\}_{m=1,2,3,\dots}$ possesses in $\mathfrak{D}$ the attribute of uniform convergence in the wider sense. Consequently the limit function $\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}f_{1\alpha}, f'_{1\alpha})$ is regular in $\mathfrak{D}$. For an arbitrary pair of $f_{1\alpha}\in\mathfrak{M}_1$ and $f'_{1\alpha}\in\mathfrak{M}_1$ satisfying (50) we set

$$U(\lambda)=R(\lambda)+\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}f_{1\alpha}, f'_{1\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta}, g'_{j\beta}), \tag{51}$$

where $R(\lambda)$, N_j, $g_{j\beta}$, and $g'_{j\beta}$ are the same notations as before. Clearly $U(\lambda)$ is regular at any point ($\neq\infty$) not belonging to the set $\overline{\{\lambda_\nu\}}\cup\left[\bigcup_{j=2}^{n}D_j\right]$ by virtue of the hypotheses concerning $g_{j\beta}$, $g'_{j\beta}$, and D_j [1]. Since there exist uncountably many functions $U(\lambda)$ with this property for fixed $\{\lambda_\nu\}$, N_j, and D_j ($j=1, 2, 3, \cdots, n$), we shall denote by $\mathfrak{F}^*$ the family of these functions. Moreover we shall call any $\lambda_\nu\in\{\lambda_\nu\}$ "an essential singularity of $U(\lambda)\in\mathfrak{F}^*$ in the sense of the functional analysis", though λ_ν is not an essential singularity of $U(\lambda)$ in the usual sense of the classical theory of functions for the case where λ_ν is one of accumulating points of $\{\lambda_\nu\}$.

Theorem 60. Let $U(\lambda)\in\mathfrak{F}^*$ be the function defined by (51); let $\{\lambda_\nu\}$ be everywhere dense on an open rectifiable curve Γ; let δ_ξ be any positive number less than the distance from an arbitrarily given point $\xi\in\Gamma$ to the set $\bigcup_{j=2}^{n}D_j$; let $\mathfrak{D}_\xi$ be the domain $\{\lambda: |\lambda-\xi|<\delta_\xi\}$; and let Δ_ξ be the domain $\mathfrak{D}_\xi-[\mathfrak{D}_\xi\cap\Gamma]$. Then $U(\lambda)$ assumes in Δ_ξ every finite value, with the possible exception of at most one finite value, an infinite number of times.

Proof. Let the two extremities of Γ be A and B; let M_1 be the middle point of the segment $\widehat{A\xi}$ of Γ; and let M_2 be the middle point of the segment $\widehat{M_1\xi}$ of Γ. By continuing this procedure we have an infinite sequence of points M_μ ($\mu=1, 2, 3, \cdots)\in\widehat{A\xi}$ tending to ξ. Similarly we construct another infinite sequence of points M'_μ ($\mu=1, 2, 3, \cdots)\in\widehat{B\xi}$ tending to ξ. We denote by p_κ the least positive integer of ν in λ_ν belonging to the set $\{\lambda_\nu\}_{\nu\geqq p}\cap\widehat{M_{\kappa-1}M_\kappa}$ where p is a positive integer. Setting $\kappa=1, 2, 3, \cdots$ and $M_0=A$, we have an infinite sequence of points λ_{p_κ} ($\kappa=1, 2, 3, \cdots)\in\widehat{A\xi}$ tending to ξ. In a similar manner, we construct another infinite sequence of points $\lambda_{p'_\kappa}$($\kappa=1, 2, 3, \cdots)\in\widehat{B\xi}$ tending to ξ. If we now consider the function

$$U_p(\lambda)=R(\lambda)+\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}f_{1\alpha}^{(p)},\, f'_{1\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta},\, g'_{j\beta})$$

where

$$f_{1\alpha}^{(p)}=\sum_{\nu=1}^{p-1}a_{1\alpha}^{(\nu)}\varphi_{\nu}^{(1)}+\sum_{\kappa=1}^{\infty}a_{1\alpha}^{(p_\kappa)}\varphi_{p_\kappa}^{(1)}+\sum_{\kappa=1}^{\infty}a_{1\alpha}^{(p'_\kappa)}\varphi_{p'_\kappa}^{(1)}\in\mathfrak{M}_1,$$

then, for any point λ not belonging to the closure $\overline{\{\lambda_\nu\}}$, we have

$$\begin{aligned} |\, U(\lambda)-U_p(\lambda)\,| &= \left|\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}-f_{1\alpha}^{(p)}),\, f'_{1\alpha})\right| \\ &= \left|\sum_{\alpha=1}^{\infty}\int_{\{\lambda_\nu\}_{\nu\leqq p}-\{\lambda_{p_\kappa},\lambda_{p'_\kappa}\}_{\kappa\geqq 1}}\frac{1}{(\lambda-z)^{\alpha}}\,d(K^{(1)}(z)\,(f_{1\alpha}-f_{1\alpha}^{(p)}),\, f'_{1\alpha})\right| \\ &\leqq \sum_{\alpha=1}^{\infty}\sum\nolimits_{p\leqq\nu\neq p_\kappa,p'_\kappa(\kappa=1,2,3,\cdots)}\frac{|\,a_{1\alpha}^{(\nu)}\overline{a}_{1\alpha}^{(\nu)}\,|}{|\,\lambda-\lambda_\nu\,|^{\alpha}}<+\infty, \end{aligned}$$

where the right-hand double series converges to 0 in accordance with (50) as p becomes infinite. This result and the expressions of $U(\lambda)$ and $U_p(\lambda)$ permit us to assert that $U(\lambda)$ is the limit function of $U_p(\lambda)$ in the entire complex λ-plane as p becomes infinite. Since, of course, the above inequality holds good in a simply connected (closed) domain E_ξ assigned arbitrarily in Δ_ξ, there exists a suitably large positive integer G such that the inequality $|\,U(\lambda)-U_p(\lambda)\,|<\varepsilon$ holds over E_ξ for an arbitrarily given positive ε and every positive integer p greater than G. Thus the infinite sequence $\{U_p(\lambda)\}_{p>G}$ converges uniformly to $U(\lambda)$ in E_ξ; and in addition, as is seen from the earlier discussion on the limit function of $\{T_m(\lambda)\}$, any $U_p(\lambda)$ is regular over E_ξ. Accordingly there exists a large positive integer G' exceeding G such that, for any complex number ω and any integer p greater than G', $U(\lambda)$, and $U_p(\lambda)$ have the same number (inclusive of 0) of ω-points in the interior of E_ξ according to a well-known theorem. This final result is also valid in Δ_ξ by virtue of the supposition that E_ξ is an arbitrary closed domain in the open domain Δ_ξ. Since, on the other hand, any $U_p(\lambda)$ with $p>G'$ has in the domain $\mathfrak{D}'_\xi\{\lambda\colon 0<|\,\lambda-\xi\,|<\delta_\xi\}$ a countably infinite number of isolated essential singularities

$$S_{p,\xi}\equiv[\{\lambda_\nu\}_{\nu=1,2,3,\cdots,p-1}\cup\{\lambda_{p_\kappa},\,\lambda_{p'_\kappa}\}_{\kappa=1,2,3,\cdots}]\cap\mathfrak{D}'_\xi$$

in the sense of the classical theory of functions, it assumes in $\mathfrak{D}'_\xi$ every value, with the possible exception of at most two values, an infinite number of times in accordance with Picard's theorem. Here it is obvious from the regularity of $U_p(\lambda)$ in the domain $\mathfrak{D}'_\xi-S_{p,\xi}$ that for $\mathfrak{D}'_\xi-S_{p,\xi}$ the point at infinity is an exceptional value of $U_p(\lambda)$. In addition, each point of $S_{p,\xi}$ is an accumulating point of ω-points of $U_p(\lambda)$ provided that ω is not an exceptional value of $U_p(\lambda)$ for its neighborhood. If we now denote by Δ a bounded domain which contains Γ in the interior of itself but not any point of $\bigcup_{j=2}^{n} D_j$, then $U(\lambda)$ is regular in $\Delta - \Gamma$. Since, however, $a_{1\alpha}^{(\nu)}\overline{\overline{a}_{1\alpha}^{(\nu)}} \neq 0$ $(\alpha, \nu = 1, 2, 3, \cdots)$ by hypothesis, $f_{1\alpha}$ and $f'_{1\alpha}$ both consist of all $\varphi_\nu^{(1)}$ and therefore it is at once obvious that not only any $\lambda_\nu \in \{\lambda_\nu\}$ is an essential singularity of $U(\lambda)$ in the sense of the functional analysis but that also every point on Γ is a singularity of $U(\lambda)$ by virtue of the hypothesis on $\{\lambda_\nu\}$. As a result, any ω-point of $U(\lambda)$ in $\mathfrak{D}'_\xi$ never lies on Γ. Again let ω be a non-exceptional value of $U_p(\lambda)$ with $p > G'$ for $\mathfrak{D}'_\xi$ and then let us suppose, contrary to what we wish to prove, that the number of ω-points of $U(\lambda)$ in Δ_ξ is finite. Then the number of ω-points of the $U_p(\lambda)$ in Δ_ξ would also be finite; for the numbers of ω-points of these two functions $U(\lambda)$ and $U_p(\lambda)$ with $p > G'$ are identical in Δ_ξ, as pointed out before. Accordingly the function $U_p(\lambda)$ would have on Γ infinitely many ω-points. Remembering that $U(\lambda)$ is the limit function of $U_p(\lambda)$ in the entire complex λ-plane as p becomes infinite and that every point on Γ is a singularity of $U(\lambda)$, the just derived result is absurd. Consequently the number of ω-points of $U(\lambda)$ in Δ_ξ is never finite.

The proof of the theorem is thus complete.

Remark 1. The hypothesis that all the D_j $(j=1, 2, 3, \cdots, n)$ lie on the disc $|\lambda| \leqq \sup_\nu |\lambda_\nu|$ is not necessary for the validity of Theorem 60, though we adopted it for the sake of later studies.

Remark 2. More generally, the result of Theorem 60 holds for the case where the set $\{\lambda_\nu\}$ is everywhere dense on a finite number of open or closed rectifiable curves, as will be verified by small modifications of the method used above.

By the same reasoning as that applied to deduce Theorem 42 from Theorem 41 [2], we can establish.

Theorem 61. Let $U(\lambda)$, Γ, ξ, δ_ξ, and Δ_ξ be the same notations as those in the statement of Theorem 60 respectively, and Δ an open domain covered by Δ_ξ as ξ ranges over Γ. Then $U(\lambda)$ has at most one finite exceptional value (in the sense of Picard) with respect to Δ.

References

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[2] ——: Some applications of the functional-representations of normal operators in Hilbert spaces. XV. Proc. Japan Acad., **41**, 150-154 (1965).