167. A Perturbation Theorem for Contraction Semi-Groups

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1. Let A be the infinitesimal generator of a contraction semi-group $\{T(\xi; A); \xi \geqq 0\}$ of class (C_0) on a Banach space X. It is well known that

(i) A is a closed linear operator and its domain $D(A)$ is dense in X,

(ii) the spectrum of A is located in the half plane $\Re(\lambda) \leqq 0$ and $\| \sigma R(\sigma + i\tau; A) \| \leqq 1$ for $\sigma > 0$, where $R(\sigma + i\tau; A)$ is the resolvent of A.

Let B likewise be the infinitesimal generator of another contraction semi-group $\{T(\xi; B); \xi \geqq 0\}$ of class (C_0) on X. Recently K. Yosida [7] proved that (i′) if $D(B) \supset D(\hat{A}_\alpha)$, where $\hat{A}_\alpha (0 < \alpha < 1)$ is the fractional power of A, then $A + B$ defined on $D(A)$ generates a contraction semi-group of class (C_0), and (ii′) if, moreover, $\{T(\xi; A); \xi \geqq 0\}$ is a holomorphic semi-group, then $A + B$ defined on $D(A)$ generates a holomorphic semi-group.

In this note we shall prove the following theorem.

Theorem. *Let $0 < \alpha < 1$ and let $\hat{B}_\alpha$ be the fractional power of B.*

(I) *Let us assume that $D(B) \supset D(A)$. Then $A + \hat{B}_\alpha$ defined on $D(A)$ generates a contraction semi-group of class (C_0).*

(II) *Assume, moreover, that $\{T(\xi; A); \xi \geqq 0\}$ is a holomorphic semi-group, then $A + \hat{B}_\alpha$ defined on $D(A)$ also generates a holomorphic semi-group.*

2. Let B be a closed linear operator with domain $D(B)$ and range in a Banach space X. Let each positive λ belong to the resolvent set of B and let

(1) $$\sup_{\lambda > 0} \| \lambda R(\lambda; B) \| = M < \infty.$$

For $0 < \alpha < 1$, the fractional power $\hat{B}_\alpha = -(-B)^\alpha$ of B is defined as follows:

(2) $$J^\alpha x = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha - 1} R(\lambda; B)(-Bx)\, d\lambda \qquad \text{for } x \in D(B),$$

(3) $\hat{B}_\alpha =$ the smallest closed linear extension of $(-J^\alpha)$.

(See [1], [2], [5], and [6]). Let likewise A be a closed linear operator with domain $D(A)$ and range in X such that "its resolvent set" $\supset\langle 0, \infty\rangle$ and $\sup_{\lambda>0}\|\lambda R(\lambda; A)\|=M'<\infty$. Then we obtain the following lemma.

Lemma. *Let us assume that $D(B)\supset D(A)$. Then for each $0<\alpha<1$, $A+\hat{B}_\alpha$ defined on $D(A)$ is a closed linear operator, and there exists a $\lambda_0>0$ such that each $\lambda\geqq\lambda_0$ belongs to the resolvent set of $A+\hat{B}_\alpha$ and $\sup_{\lambda\geqq\lambda_0}\|\lambda R(\lambda; A+\hat{B}_\alpha)\|<\infty$.*

Proof. By (1),

$$\|\hat{B}_\alpha x\|\leqq\frac{\sin\alpha\pi}{\pi}\int_0^\infty\lambda^{\alpha-1}\|R(\lambda; B)Bx\|\,d\lambda$$
$$\leqq\frac{\sin\alpha\pi}{\pi}\left[\int_0^w\lambda^{\alpha-1}\|\lambda R(\lambda; B)x-x\|\,d\lambda+\int_w^\infty\lambda^{\alpha-1}\|R(\lambda; B)Bx\|\,d\lambda\right]$$
$$\leqq(M+1)\,w^\alpha\frac{\sin\alpha\pi}{\alpha\pi}\|x\|+\frac{M\sin\alpha\pi}{w^{1-\alpha}\pi(1-\alpha)}\|Bx\|$$

for $x\in D(B)$, where w is an arbitrary positive number. The condition $D(B)\supset D(A)$ implies, by the closed graph theorem, that there exists a constant $K>0$ such that

$$\|Bx\|\leqq K(\|Ax\|+\|x\|)\qquad\text{for } x\in D(A).$$

Thus we have

$$\|\hat{B}_\alpha x\|\leqq C_w\|Ax\|+C'_w\|x\|\qquad\text{for } x\in D(A),$$

where

$$C_w=\frac{KM\sin\alpha\pi}{\pi(1-\alpha)w^{1-\alpha}}\quad\text{and}\quad C'_w=\frac{(M+1)w^\alpha\sin\alpha\pi}{\alpha\pi}+\frac{KM\sin\alpha\pi}{\pi(1-\alpha)w^{1-\alpha}}.$$

Then

$$\|\hat{B}_\alpha R(\lambda; A)x\|\leqq C_w\|AR(\lambda; A)x\|+C'_w\|R(\lambda; A)x\|$$
$$\leqq[(M'+1)C_w+M'C'_w\lambda^{-1}]\,\|x\|$$

for $x\in X$ and $\lambda>0$. We take $w_0>0$ such that $(M'+1)C_{w_0}<1/2$, and we put $\lambda_0=2M'C'_{w_0}$. Then we have

$$\|\hat{B}_\alpha R(\lambda; A)\|\leqq\beta\qquad\text{for } \lambda\geqq\lambda_0,$$

where $\beta=(M'+1)C_{w_0}+1/2<1$. This proves that, for each $\lambda\geqq\lambda_0$, the inverse $(I-\hat{B}_\alpha R(\lambda; A))^{-1}=\sum_{n=0}^\infty[\hat{B}_\alpha R(\lambda; A)]^n$ exists as a bounded linear operator from X onto itself and $\|[I-\hat{B}_\alpha R(\lambda; A)]^{-1}\|\leqq(1-\beta)^{-1}$. Since $\lambda-(A+\hat{B}_\alpha)=[I-\hat{B}_\alpha R(\lambda; A)]\,(\lambda-A)$, we obtain that each $\lambda\geqq\lambda_0$ belongs to the resolvent set of $A+\hat{B}_\alpha$ and $R(\lambda;A+\hat{B}_\alpha)=R(\lambda;A)[I-\hat{B}_\alpha R(\lambda;A)]^{-1}$. Then $A+\hat{B}_\alpha$ is a closed linear operator and $\|\lambda R(\lambda; A+\hat{B}_\alpha)\|\leqq\|\lambda R(\lambda; A)\|\cdot\|[I-\hat{B}_\alpha R(\lambda; A)]^{-1}\|\leqq M'(1-\beta)^{-1}$ for $\lambda\geqq\lambda_0$. This completes the proof.

3. Proof of the Theorem. We first remark that $\hat{B}_\alpha$ generates a contraction semi-group of class (C_0). Then A and $\hat{B}_\alpha$ are both dissipative in the sense of G. Lumer and R. S. Phillips [3]. That is, if we take $\varphi_x\in X^*$ for $x\in X$ such that $\|\varphi_x\|=\|x\|$ and $(x, \varphi_x)=\|x\|^2$, then we have

$$\Re(Ax, \varphi_x)\leqq 0\quad\text{for}\quad x\in D(A)\quad\text{and}\quad\Re(\hat{B}_\alpha x, \varphi_x)\leqq 0\quad\text{for}\quad x\in D(\hat{B}_\alpha).$$

Therefore $\Re([\lambda-(A+\hat{B}_\alpha)]x, \varphi_x)\geqq\lambda\|x\|^2$ for $x\in D(A)$ and $\lambda>0$, so that we have

(4) $\quad \|[\lambda-(A+\hat{B}_\alpha)]x\|\geqq\lambda\|x\| \qquad$ for $x\in D(A)$ and $\lambda>0$.

Since A and B satisfy the conditions of Lemma, $A+\hat{B}_\alpha$ defined on $D(A)$ is a closed linear operator and sufficiently large $\lambda>0$ belongs to its resolvent set. Then, by (4), we have

$$\|\lambda R(\lambda; A+\hat{B}_\alpha)\|\leqq 1$$

for sufficiently large $\lambda>0$. And since $D(A+\hat{B}_\alpha)=D(A)$ is dense in X, the Hille-Yosida theorem shows that $A+\hat{B}_\alpha$ generates a contraction semi-group of class (C_0).

Furthermore, if $\{T(\xi; A); \xi\geqq 0\}$ is a holomorphic semi-group,[1] then we have

(5) $\quad \overline{\lim_{|\tau|\to\infty}}\|\tau R(\sigma+i\tau; A)\|<\infty \qquad$ for $\sigma>0$.

It follows, from $\|\sigma R(\sigma+i\tau; A)\|\leqq 1$ and (5), that

$$\sup_{-\infty<\tau<\infty}\|\tau R(\sigma+i\tau; A)\|=K_\sigma<\infty$$

for $\sigma>0$. Especially, for fixed $\sigma_0>0$,

$$\sup_{-\infty<\tau<\infty}\|\tau R(\sigma_0+i\tau; A)\|=K_{\sigma_0}<\infty.$$

By the resolvent equation

$$R(\sigma+i\tau; A)=R(\sigma_0+i\tau; A)-(\sigma-\sigma_0)R(\sigma+i\tau; A)R(\sigma_0+i\tau; A),$$

we get

$$\|\tau R(\sigma+i\tau; A)\|\leqq K_{\sigma_0}(1+|1-\sigma_0/\sigma|)$$

for $\sigma>0$ and τ; especially if $\sigma\geqq\sigma_0$, then

$$(6)\quad \begin{cases} \|\tau R(\sigma+i\tau; A)\|\leqq 2K_{\sigma_0} \text{ and} \\ \|(\sigma+i\tau)R(\sigma+i\tau; A)\|\leqq\|\sigma R(\sigma+i\tau; A)\|+\|\tau R(\sigma+i\tau; A)\| \\ \qquad\leqq 2K_{\sigma_0}+1 \end{cases}$$

for all τ.

Hence, similarly as in the proof of Lemma, we have $\|\hat{B}_\alpha R(\sigma+i\tau; A)\|\leqq\beta'(<1)$ for sufficiently large $\sigma>0$ and for all τ, where β' is a constant independent of σ and τ, so that the inverse

$$[I-\hat{B}_\alpha R(\sigma+i\tau; A)]^{-1}=\sum_{n=0}^{\infty}[\hat{B}_\alpha R(\sigma+i\tau; A)]^n$$

exists and

(7) $\quad \|[I-\hat{B}_\alpha R(\sigma+i\tau; A)]^{-1}\|\leqq(1-\beta')^{-1}$.

Since $R(\sigma+i\tau; A+\hat{B}_\alpha)=R(\sigma+i\tau; A)[I-\hat{B}_\alpha R(\sigma+i\tau; A)]^{-1}$, by (6) and (7), we have

(8) $\quad \sup_{-\infty<\tau<\infty}\|\tau R(\sigma+i\tau; A+\hat{B}_\alpha)\|\leqq 2K_{\sigma_0}(1-\beta')^{-1}$

for sufficiently large $\sigma>0$. We already proved that $A+\hat{B}_\alpha$ generates a contraction semi-group $\{T(\xi; A+\hat{B}_\alpha); \xi\geqq 0\}$ of class (C_0). Thus the above inequality (8) shows that $\{T(\xi; A+\hat{B}_\alpha); \xi\geqq 0\}$ is a holomorphic semi-group. This completes the proof.

Remark. Let X be a Banach lattice and let A be the in-

1) See [6].

finitesimal generator of a semi-group of class (C_0) of positive contraction operators on X. Let likewise B be the infinitesimal generator of another semi-group of class (C_0) of positive contraction operators on X. Then we can prove that *if* $D(B)\supset D(A)$, *then* $A+\hat{B}_\alpha(0<\alpha<1)$ *generates a semi-group of class* (C_0) *of positive contraction operators.* In fact, since $\hat{B}_\alpha$ also generates a semi-group of class (C_0) of positive contraction operators, A and $\hat{B}_\alpha$ are both dispersive in the sense of R. S. Phillips [4]. Then $A+\hat{B}_\alpha$ defined on $D(A)$ is also a dispersive operator. We already proved, in Theorem, that $A+\hat{B}_\alpha$ is the infinitesimal generator of a contraction semi-group of class (C_0), and hence "the range of $I-(A+\hat{B}_\alpha)$"$=X$. Thus it follows from the Phillips theorem [4, Theorem 2.1] that $A+\hat{B}_\alpha$ generates a semi-group of class (C_0) of positive contraction operators.

References

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