166. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XXIII

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Let D_j ($j=1$ to n), $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$, N_j ($j=1$ to n), $f_{1\alpha}$, $f_{2\alpha}$, $f'_{1\alpha}$, $f'_{2\alpha}$, $g_{j\beta}$, and $g'_{j\beta}$ be the same notations as those defined in Part XIII [cf. Proc. Japan Acad., Vol. 40, pp. 492–497 (1964)], and $R(\lambda)$ an integral function. Throughout this paper we deal with a resolvent function $\tilde{U}(\lambda)$ concerning the bounded normal operators N_j such that

$$\begin{aligned}\tilde{U}(\lambda)&=R(\lambda)+\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}(f_{1\alpha}+f_{2\alpha}),\ f'_{1\alpha}+f'_{2\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta},\ g'_{j\beta})\\&=R(\lambda)+\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}c_\alpha^{(\nu)}(\lambda-\lambda_\nu)^{-\alpha}+\sum_{\alpha=1}^{\infty}\int_{[\overline{\{\lambda_\nu\}}-\{\lambda_\nu\}]\cup D_1}(\lambda-\zeta)^{-\alpha}d(K^{(1)}(\zeta)f_{2\alpha},\ f'_{2\alpha})\\&\quad+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}\int_{D_j}(\lambda-\zeta)^{-\beta}d(K^{(j)}(\zeta)g_{j\beta},\ g'_{j\beta})\end{aligned}$$

where $\{K^{(j)}(\zeta)\}$ denotes the complex spectral family of N_j for each value of $j=1, 2, 3, \cdots, n$, on the assumptions that

$$\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}|c_\alpha^{(\nu)}(\lambda-\lambda_\nu)^{-\alpha}|<\infty\qquad(\lambda\notin\overline{\{\lambda_\nu\}})$$

and

$$\sum_{\alpha=1}^{\infty}\left|\int_{[\overline{\{\lambda_\nu\}}-\{\lambda_\nu\}]\cup D_1}(\lambda-\zeta)^{-\alpha}d(K^{(1)}(\zeta)f_{2\alpha},\ f'_{2\alpha})\right|<\infty\qquad(\lambda\notin\overline{\{\lambda_\nu\}}\cup D_1).$$

In fact, as will be seen from the method used to show that there exist uncountably many pairs of $f_{1\alpha}$ and $f'_{1\alpha}$ such that the former inequality holds [cf. Proc. Japan Acad., Vol. 42, pp. 583–588 (1966)], we can find uncountably many pairs of $f_{2\alpha}$ and $f'_{2\alpha}$ such that the latter inequality holds.

Theorem 64. Let $\tilde{U}(\lambda)$ be the function defined above, and let $\overline{\{\lambda_\nu\}}\cup\left[\bigcup_{j=1}^{n}D_j\right]$ be contained in the disc $\overline{\mathfrak{D}}_\sigma\{\lambda: |\lambda|\leqq\sigma\}$. Then $\tilde{U}(\lambda)$ is expansible on any domain $\varDelta_\rho\{\lambda: \rho<|\lambda|<\infty\}$ with $\sigma<\rho<\infty$ in the form

$$\tilde{U}\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)=\frac{1}{2}a_0+\frac{1}{2}\sum_{p=1}^{\infty}(a_p-ib_p)\Big(\frac{e^{i\theta}}{\kappa}\Big)^p+\frac{1}{2}\sum_{p=1}^{\infty}(a_p+ib_p)\Big(\frac{\kappa}{e^{i\theta}}\Big)^p\qquad(0<\kappa<1)$$

where

$$\left.\begin{aligned}a_p&=\frac{1}{\pi}\int_0^{2\pi}\tilde{U}(\rho e^{it})\cos ptdt\\b_p&=\frac{1}{\pi}\int_0^{2\pi}\tilde{U}(\rho e^{it})\sin ptdt\end{aligned}\right\}\qquad(p=0, 1, 2, \cdots)\tag{52}$$

and the two series on the right converge absolutely and uniformly for any κ with $0<\kappa<1$. Moreover the ordinary part $R(\lambda)$ and the sum-function $\chi(\lambda)$ of the first and second principal parts of $\tilde{U}(\lambda)$ are expansible in the forms

$$R(\kappa\rho e^{i\theta})=\frac{1}{2}a_0+\frac{1}{2}\sum_{p=1}^{\infty}(a_p-ib_p)(\kappa e^{i\theta})^p \qquad (0\leqq\kappa<\infty)$$

and

$$\chi\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)=\frac{1}{2}\sum_{p=1}^{\infty}(a_p+ib_p)\Big(\frac{\kappa}{e^{i\theta}}\Big)^p \qquad (0<\kappa<1)$$

respectively.

Proof. Since this theorem can be established by reasoning exactly like that applied to obtain the expansion of the function $S(\lambda)$ or $T(\lambda)$ treated in the preceding papers, we will only give an outline of the proof here.

In the interests of brevity, we shall put

$$\Phi(\lambda)=\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}f_{1\alpha},\, f'_{1\alpha}),$$

$$\Psi(\lambda)=\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}f_{2\alpha},\, f'_{2\alpha})+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}((\lambda I-N_j)^{-\beta}g_{j\beta},\, g'_{j\beta}).$$

Then $\Phi(\lambda)$ and $\Psi(\lambda)$ are the first principal part and the second principal part of $\tilde{U}(\lambda)$ respectively and so $\chi(\lambda)=\Phi(\lambda)+\Psi(\lambda)$. We now denote by Γ an arbitrarily given closed Jordan curve containing $\overline{\{\lambda_\nu\}}\cup\Big[\bigcup_{j=1}^{n}D_j\Big]$ inside itself. Since, by assumptions, $\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}c_\alpha^{(\nu)}(\lambda-\lambda_\nu)^{-\alpha}$ is absolutely and uniformly convergent in any closed domain $\Delta_\rho\{\lambda\colon \rho\leqq|\lambda|\}$ with $\sigma<\rho<\infty$, we can find with the aid of the Cauchy theorem and the calculus of residues that, if Γ is positively oriented,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma \Phi(\lambda)(\lambda-z)^{-1}d\lambda\\
&=\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{1}{2\pi i}\int_\Gamma c_\alpha^{(\nu)}(z-\lambda_\nu)^{-1}\{(\lambda-z)^{-1}(\lambda-\lambda_\nu)^{-\alpha+1}-(\lambda-\lambda_\nu)^{-\alpha}\}d\lambda\\
&=\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{1}{2\pi i}\int_\Gamma c_\alpha^{(\nu)}(z-\lambda_\nu)^{-2}\{(\lambda-z)^{-1}(\lambda-\lambda_\nu)^{-\alpha+2}-(\lambda-\lambda_\nu)^{-\alpha+1}\}d\lambda\\
&\;\vdots\\
&=\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\frac{1}{2\pi i}\int_\Gamma c_\alpha^{(\nu)}(z-\lambda_\nu)^{-\alpha}\{(\lambda-z)^{-1}-(\lambda-\lambda_\nu)^{-1}\}d\lambda,\\
&=\begin{cases}0 & \text{(for every } z \text{ inside } \Gamma)\\ -\Phi(z) & \text{(for every } z \text{ outside } \Gamma)\end{cases}
\end{aligned}$$

because of the fact that

$$\frac{1}{2\pi i}\int_\Gamma c_\alpha^{(\nu)}(\lambda-\lambda_\nu)^{-\alpha-1}d\lambda=0$$

for the term $c_\alpha^{(\nu)}(\lambda-\lambda_\nu)^{-\alpha-1}$ appearing in the expansion of $\Phi(\lambda)(\lambda-\lambda_\nu)^{-1}$.

Since, on the other hand, $\sum_{\alpha=1}^{\infty}((\lambda I-N_1)^{-\alpha}f_{2\alpha}, f'_{2\alpha})$ also converges absolutely and uniformly in $\bar{\Delta}_\rho$ by virtue of the assumptions, we have

$$\frac{1}{2\pi i}\int_\Gamma \Psi(\lambda)(\lambda-z)^{-1}d\lambda$$

$$=\sum_{\alpha=1}^{\infty}\int_{[\overline{\{\lambda_\nu\}}-\{\lambda_\nu\}]\cup D_1}\frac{1}{2\pi i}\int_\Gamma(\lambda-z)^{-1}(\lambda-\zeta)^{-\alpha}d\lambda d(K^{(1)}(\zeta)f_{2\alpha}, f'_{2\alpha})$$

$$+\sum_{j=2}^{n}\sum_{\beta=1}^{k_j}\int_{D_j}\frac{1}{2\pi i}\int_\Gamma(\lambda-z)^{-1}(\lambda-\zeta)^{-\beta}d\lambda d(K^{(j)}(\zeta)g_{j\beta}, g'_{j\beta});$$

and moreover, supposing that ζ belongs to $[\overline{\{\lambda_\nu\}}-\{\lambda_\nu\}]\cup D_1$ or to D_j according as m is equal to α or to β, we have

$$\frac{1}{2\pi i}\int_\Gamma(\lambda-z)^{-1}(\lambda-\zeta)^{-m}d\lambda=\frac{1}{2\pi i}\int_\Gamma(z-\zeta)^{-m}\{(\lambda-z)^{-1}-(\lambda-\zeta)^{-1}\}d\lambda$$

$$=\begin{cases}0 & \text{(for every } z \text{ inside } \Gamma)\\ -(z-\zeta)^{-m} & \text{(for every } z \text{ outside } \Gamma),\end{cases}$$

as is seen from the fact that the left-hand side vanishes for $z=\zeta$. Consequently

$$\frac{1}{2\pi i}\int_\Gamma \Psi(\lambda)(\lambda-z)^{-1}d\lambda=\begin{cases}0 & \text{(for every } z \text{ inside } \Gamma)\\ -\Psi(z) & \text{(for every } z \text{ outside } \Gamma).\end{cases}$$

These results imply that

$$\frac{1}{2\pi i}\int_\Gamma \chi(\lambda)(\lambda-z)^{-k}d\lambda=\begin{cases}0 & \text{(for every } z \text{ inside } \Gamma)\\ -\chi^{(k-1)}(z)/(k-1)! & \text{(for every } z \text{ outside } \Gamma)\end{cases}$$

and hence that

$$\text{(53)}\quad \frac{1}{2\pi i}\int_\Gamma \tilde{U}(\lambda)(\lambda-z)^{-k}d\lambda=R^{(k-1)}(z)/(k-1)! \quad \text{(for every } z \text{ inside } \Gamma).$$

By making use of the relation $\frac{1}{2}(a_p-ib_p)=R^{(p)}(0)\rho^p/p!$ $(\sigma<\rho<\infty)$ derived from $\frac{1}{2\pi i}\int_\Gamma \tilde{U}(\lambda)\lambda^{-p-1}d\lambda=R^{(p)}(0)/p!$, we can first establish the equality

$$R(\kappa\rho e^{i\theta})=\frac{1}{2}a_0+\frac{1}{2}\sum_{p=1}^{\infty}(a_p-ib_p)(\kappa e^{i\theta})^p \qquad (0\leqq\kappa<\infty)$$

[cf. Proc. Japan Acad., Vol. 38, pp. 641–645 (1962)]. Next it is verified with the help of (53) that

$$\chi\Big(\frac{\lambda\bar{\lambda}}{\bar{z}}\Big)+R(z)=\frac{1}{2\pi}\int_0^{2\pi}\tilde{U}(\lambda)\Re[(\lambda+z)(\lambda-z)^{-1}]dt$$

$$(|z|<\rho,\ \sigma<\rho<\infty,\ \lambda=\rho e^{it})$$

[cf. Proc. Japan Acad., Vol. 38, pp. 452–456 (1962)]. On setting $z=re^{i\theta}$, we have therefore

$$\tilde{U}\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)-R\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)+R(\kappa\rho e^{i\theta})$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\tilde{U}(\rho e^{it})(1-\kappa^2)[1+\kappa^2-2\kappa\cos(\theta-t)]^{-1}dt \qquad (0<\kappa<1),$$

where the right-hand side is expansible in the form

$$\frac{1}{2}a_0+\sum_{p=1}^{\infty}\kappa^p(a_p\cos p\theta+b_p\sin p\theta).$$

From the final result we can derive the desired equality in the statement of the theorem. Moreover the absolute and uniform convergency of the expansion of $\tilde{U}\left(\frac{\rho}{\kappa}e^{i\theta}\right)$ for any κ with $0<\kappa<1$ is found from the facts that the sets $\{a_p\}$ and $\{b_p\}$ are both bounded and the equalities $\frac{1}{2}(a_p-ib_p)=R^{(p)}(0)\rho^p/p!$ $(p=0, 1, 2, \cdots)$ are valid.

Remark. Clearly, by virtue of the results of Theorem 64, all the propositions deduced from the expansion of each of $S(\lambda)$ and $T(\lambda)$ in the earlier discussions are true of $\tilde{U}(\lambda)$. In addition, most of other propositions established for $T(\lambda)$ are also valid for $\tilde{U}(\lambda)$, as will be seen from the methods of their proofs. It must, however, be noted that there are some essential differences between the respective characteristics of $T(\lambda)$ and $\tilde{U}(\lambda)$ as we indicated in the preceding papers.

Theorem 65. Let $\tilde{U}(\lambda)$ be the function in Theorem 64; let $\{\lambda'_\nu\}_{\nu=1,2,3,\dots}$ be an arbitrarily prescribed, bounded, and infinite sequence of complex numbers; let D'_j ($j=1$ to n') be mutually disjoint, closed, bounded, and connected domains having no point in common with the closure $\overline{\{\lambda'_\nu\}}$ of $\{\lambda'_\nu\}$; let N'_j be a bounded normal operator whose point spectrum and continuous spectrum are given by $\{\lambda'_\nu\}$ and $[\overline{\{\lambda'_\nu\}}-\{\lambda'_\nu\}]\cup D'_j$ respectively for each value of $j=1, 2, \cdots, n'$; let $\hat{f}_{1\alpha}$ and $\hat{f}'_{1\alpha}$ be elements of the subspace $\mathfrak{M}'_1$ determined by all mutually orthogonal normalized eigenelements of N'_1; let $\hat{f}_{2\alpha}$ and $\hat{f}'_{2\alpha}$ be elements of the orthogonal complement $\mathfrak{N}'_1$ of $\mathfrak{M}'_1$ in the complex abstract Hilbert space $\mathfrak{H}$ under consideration; let $\hat{g}_{j\beta}$ and $\hat{g}'_{j\beta}$ be elements in the subspace $\hat{K}_j(D'_j)\mathfrak{H}$ where $\{\hat{K}_j(\lambda)\}$ denotes the complex spectral family of N'_j; let $\hat{R}(\lambda)$ be an integral function; and let

$$\hat{U}(\lambda)=\hat{R}(\lambda)+\sum_{\alpha=1}^{\infty}((\lambda I-N'_1)^{-\alpha}\hat{f}_{1\alpha},\hat{f}'_{1\alpha})+\sum_{\alpha=1}^{\infty}((\lambda I-N'_1)^{-\alpha}\hat{f}_{2\alpha},\hat{f}'_{2\alpha})$$
$$+\sum_{j=2}^{n'}\sum_{\beta=1}^{k'_j}((\lambda I-N'_j)^{-\beta}\hat{g}_{j\beta},\hat{g}'_{j\beta})\qquad(2\leqq n'<\infty,\ 1\leqq k'_j<\infty)$$

where $\hat{f}_{1\alpha}, \hat{f}'_{1\alpha}, \hat{f}_{2\alpha}$, and $\hat{f}'_{2\alpha}$ are so chosen as to satisfy the conditions

$$\sum_{\alpha=1}^{\infty}|((\lambda I-N'_1)^{-\alpha}\hat{f}_{1\alpha},\hat{f}'_{1\alpha})|\leqq\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}|\hat{c}_\alpha^{(\nu)}(\lambda-\lambda'_\nu)^{-1}|<\infty\qquad(\lambda\notin\overline{\{\lambda'_\nu\}})$$

and

$$\sum_{\alpha=1}^{\infty}|((\lambda I-N'_1)^{-\alpha}\hat{f}_{2\alpha},\hat{f}'_{2\alpha})|<\infty\qquad(\lambda\notin\overline{\{\lambda'_\nu\}}\cup D'_1);$$

let Γ be a rectifiable closed Jordan curve containing the respective sets $\overline{\{\lambda_\nu\}}\cup[\bigcup_{j=1}^{n}D_j]$ and $\overline{\{\lambda'_\nu\}}\cup[\bigcup_{j=1}^{n'}D'_j]$ of singularities of $\tilde{U}(\lambda)$ and $\hat{U}(\lambda)$ on the complex λ-plane $\{\lambda: |\lambda|<\infty\}$ inside itself; let ρ be any positive constant such that the circle $\{\lambda: |\lambda|=\rho\}$ contains $\overline{\{\lambda_\nu\}}\cup[\bigcup_{j=1}^{n} D_j]$ and $\overline{\{\lambda'_\nu\}}\cup[\bigcup_{j=1}^{n'} D'_j]$ inside itself and does not intersect Γ; and let $K_p=a_p^2+b_p^2$ and $\hat{K}_p=\hat{a}_p^2+\hat{b}_p^2$ ($p=0, 1, 2, \cdots$) where a_p and b_p are given by (52) and

$$\left.\begin{aligned}\hat{a}_p&=\frac{1}{\pi}\int_0^{2\pi}\hat{U}(\rho e^{it})\cos pt dt\\ \hat{b}_p&=\frac{1}{\pi}\int_0^{2\pi}\hat{U}(\rho e^{it})\sin pt dt\end{aligned}\right\}\qquad (p=0, 1, 2, \cdots).$$

Then K_p and $\hat{K}_p$ ($p=1, 2, 3, \cdots$) are constants independent of ρ; and assuming that $K_{p+1}/R^{(p+1)}(0)$ denotes $\frac{1}{2\pi i}\int_\Gamma \tilde{U}(\lambda)\lambda^p d\lambda\cdot 4/(p+1)!$ when $R^{(p+1)}(0)=0$ and that $\hat{K}_{p+1}/\hat{R}^{(p+1)}(0)$ denotes $\frac{1}{2\pi i}\int_\Gamma \hat{U}(\lambda)\lambda^p d\lambda\cdot 4/(p+1)!$ when $\hat{R}^{(p+1)}(0)=0$,

$$\frac{1}{2\pi i}\int_\Gamma \tilde{U}(\lambda)\hat{U}(\lambda)d\lambda=\frac{1}{4}\Big\{\sum_{p=0}^{\infty}(p+1)R^{(p)}(0)\hat{K}_{p+1}/\hat{R}^{(p+1)}(0)$$
$$+\sum_{p=0}^{\infty}(p+1)\hat{R}^{(p)}(0)K_{p+1}/R^{(p+1)}(0)\Big\},$$

where the complex line integral around Γ is taken counterclockwise and the two series on the right are absolutely convergent.

Proof. By the definitions of Γ and the circle $\{\lambda: |\lambda|=\rho\}$, Γ does not intersect the circle $C\{\lambda: |\lambda|=\rho/\kappa\}$ for a suitable positive κ less than 1. Both $\tilde{U}(\lambda)$ and $\hat{U}(\lambda)$ are regular on the closed domain surrounded by Γ and C and so it follows from the Cauchy theorem that

$$\frac{1}{2\pi i}\int_\Gamma \tilde{U}(\lambda)\hat{U}(\lambda)d\lambda=\frac{1}{2\pi i}\int_C \tilde{U}(\lambda)\hat{U}(\lambda)d\lambda,$$

the complex line integrals around Γ and C being taken counterclockwise. Since, on the other hand,

$$\tilde{U}\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)=\frac{1}{2}a_0+\frac{1}{2}\sum_{p=1}^{\infty}(a_p-ib_p)\Big(\frac{e^{i\theta}}{\kappa}\Big)^p+\frac{1}{2}\sum_{p=1}^{\infty}(a_p+ib_p)\Big(\frac{\kappa}{e^{i\theta}}\Big)^p\qquad(0<\kappa<1)$$

and

$$\hat{U}\Big(\frac{\rho}{\kappa}e^{i\theta}\Big)=\frac{1}{2}\hat{a}_0+\frac{1}{2}\sum_{p=1}^{\infty}(\hat{a}_p-i\hat{b}_p)\Big(\frac{e^{i\theta}}{\kappa}\Big)^p+\frac{1}{2}\sum_{p=1}^{\infty}(\hat{a}_p+i\hat{b}_p)\Big(\frac{\kappa}{e^{i\theta}}\Big)^p\qquad(0<\kappa<1),$$

and since, in addition, the series on the right of each of these expansions is not only absolutely convergent but also uniformly convergent with respect to θ, we can verify by direct computation that

$$\begin{aligned}&\frac{1}{2\pi i}\int_C \tilde{U}(\lambda)\hat{U}(\lambda)d\lambda\\ &=\frac{\rho}{2\pi\kappa}\int_0^{2\pi}\tilde{U}\Big(\frac{\rho}{\kappa}e^{it}\Big)\hat{U}\Big(\frac{\rho}{\kappa}e^{it}\Big)e^{it}dt\\ &=\frac{\rho}{4}\Big\{\sum_{p=0}^{\infty}(a_p-ib_p)(\hat{a}_{p+1}+i\hat{b}_{p+1})+\sum_{p=0}^{\infty}(\hat{a}_p-i\hat{b}_p)(a_{p+1}+ib_{p+1})\Big\},\end{aligned}$$

where denoting by $\hat{\chi}(\lambda)$ the sum-function of the first and second principal parts of $\hat{U}(\lambda)$,

$$\frac{\rho}{4}\sum_{p=0}^{\infty}(a_p-ib_p)(\hat{a}_{p+1}+i\hat{b}_{p+1})=\frac{1}{2\pi i}\int_{C}R(\lambda)\hat{\chi}(\lambda)d\lambda$$

and

$$\frac{\rho}{4}\sum_{p=0}^{\infty}(\hat{a}_p-i\hat{b}_p)(a_{p+1}+ib_{p+1})=\frac{1}{2\pi i}\int_{C}\hat{R}(\lambda)\chi(\lambda)d\lambda.$$

If we now set $M(\rho)=\max_{t\in[0,2\pi]}|\tilde{U}(\rho e^{it})|$ and $\hat{M}(\rho)=\max_{t\in[0,2\pi]}|\hat{U}(\rho e^{it})|$, we have

$$\sum_{p=0}^{\infty}|a_p-ib_p|\,|\hat{a}_{p+1}+i\hat{b}_{p+1}|\leqq 4\sum_{p=0}^{\infty}\rho^p\hat{M}(\rho)\,|R^{(p)}(0)|/p!<\infty$$

and

$$\sum_{p=0}^{\infty}|\hat{a}_p-i\hat{b}_p|\,|a_{p+1}+ib_{p+1}|\leqq 4\sum_{p=0}^{\infty}\rho^pM(\rho)\,|\hat{R}^{(p)}(0)|/p!<\infty$$

in accordance with $a_p-ib_p=2R^{(p)}(0)\rho^p/p!$, $\hat{a}_p-i\hat{b}_p=2\hat{R}^{(p)}(0)\rho^p/p!$, $|a_p+ib_p|\leqq\frac{1}{\pi}\int_0^{2\pi}|\tilde{U}(\rho e^{it})|\,dt\leqq 2M(\rho)$, and $|\hat{a}_p+i\hat{b}_p|\leqq 2\hat{M}(\rho)$ $(p=0, 1, 2, \cdots)$. Since, by reasoning exactly like that used in the case of the function $S(\lambda)$ treated before [cf. Proc. Japan Acad., Vol. 38, pp. 646–650 (1962)], we can show that K_p and $\hat{K}_p$ $(p=1, 2, 3, \cdots)$ are constants independent of ρ, it remains only to prove that the equalities

$$\rho(a_p-ib_p)(\hat{a}_{p+1}+i\hat{b}_{p+1})=(p+1)R^{(p)}(0)\hat{K}_{p+1}/\hat{R}^{(p+1)}(0),$$
$$\rho(\hat{a}_p-i\hat{b}_p)(a_{p+1}+ib_{p+1})=(p+1)\hat{R}^{(p)}(0)K_{p+1}/R^{(p+1)}(0)$$

hold on the assumption that, when $R^{(p+1)}(0)$ and $\hat{R}^{(p+1)}(0)$ vanish, $K_{p+1}/R^{(p+1)}(0)$ and $\hat{K}_{p+1}/\hat{R}^{(p+1)}(0)$ have such meanings as were defined in the statement of the present theorem.

Now, suppose that $\hat{R}^{(p+1)}(0)$ is not zero. In fact, it is found immediately from the equalities $a_p-ib_p=2R^{(p)}(0)\rho^p/p!$ and $\hat{a}_{p+1}+i\hat{b}_{p+1}=(p+1)!\,\hat{K}_{p+1}/2\hat{R}^{(p+1)}(0)\rho^{p+1}$ that

$$\rho(a_p-ib_p)(\hat{a}_{p+1}+i\hat{b}_{p+1})=(p+1)R^{(p)}(0)\hat{K}_{p+1}/\hat{R}^{(p+1)}(0).$$

Next, suppose that $\hat{R}^{(p+1)}(0)$ vanishes and then that the symbol $\hat{K}_{p+1}/\hat{R}^{(p+1)}(0)$ denotes $\frac{1}{2\pi i}\int_{\Gamma}\hat{U}(\lambda)\lambda^pd\lambda\cdot 4/(p+1)!$. Then we have

$$\rho(a_p-ib_p)(\hat{a}_{p+1}+i\hat{b}_{p+1})=4R^{(p)}(0)/p!\cdot\frac{1}{2\pi i}\int_{C}\hat{U}(\lambda)\lambda^pd\lambda$$
$$=(p+1)R^{(p)}(0)\hat{K}_{p+1}/\hat{R}^{(p+1)}(0).$$

Likewise we can show the validity of the equality

$$\rho(\hat{a}_p-i\hat{b}_p)(a_{p+1}+ib_{p+1})=(p+1)\hat{R}^{(p)}(0)K_{p+1}/R^{(p+1)}(0),$$

assuming that, when $R^{(p+1)}(0)=0$, the symbol $K_{p+1}/R^{(p+1)}(0)$ denotes $\frac{1}{2\pi i}\int_{\Gamma}\tilde{U}(\lambda)\lambda^pd\lambda\cdot 4/(p+1)!$. The theorem has thus been proved.