# 154. Semigroups Connected with Equivalence and Congruence Relations

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0. *Introduction.* The idea of semiclosure operations is useful for finding the smallest equivalence or congruence relation which contains a given relation $\rho$. It is obtained by applying to $\rho$ the reflexive operation $R$, symmetric operation $S$, compatible operation $C$, and transitive operation $T$ [2], [3]. There are many combinations of operations which give the same equivalence or congruence relation, for example, $RST$ and $RTST$. Using the concept of free semigroup and defining relations we find all words which when interpreted as operations give equivalence and congruence relations; we study the structure of the semigroup generated by $R, S, T$ or $R, S, T, C$. The defining relations when interpreted as operations are identities. Some results of this paper were published in [2], [3] without proof. Before entering the main discussion we introduce the concept of annexed product or coproduct.

Let $G$ be a groupoid. By $G^1$ we mean adjoining an identity to $G$ even if $G$ already has one. The annexed product of two groupoids $A$ and $B$ is the direct product of $A^1$ and $B^1$ minus the element (1,1).

$$A \tilde{\times} B = A^1 \times B^1 - \{(1, 1)\}.$$

$G$ is isomorphic to $A \tilde{\times} B$ iff $G$ contains two subgroupoids $\hat{A}, \hat{B}$ isomorphic to $A$ and $B$ respectively such that every element of $G$ can be uniquely expressed as a product $ab$, where $a \in \hat{A}^1$ and $b \in \hat{B}^1$, and the elements of $\hat{A}$ and $\hat{B}$ commute.

1. *Equivalence-Semigroup.* In this section we study the structure of the semigroup generated by $R, S, T$, [3]. Let $Q^*$ be the semigroup generated by $R, S, T$, subject to the defining relations (1,1).

(1.1) $R^2=R,\ S^2=S,\ T^2=T,\ RS=SR,\ RT=TR,\ STS=TST=ST.$

Theorem 1.1. *$Q^*$ is composed of nine elements.*

(1.2) $R, S, T, RS, RT, ST, TS, RST, RTS.$

Proof. Since $R$ commutes with $S$ and $T$, if a word contains $R$, then it has the form $R\cdot W(S, T)$ where $W(S, T)$ is a word of $S$ and $T$. Let $W(S, T)$ be a word of $S$ and $T$ with length $n \geq 2$. By induction on $n$ we can prove $W(S, T)$ is either $ST$ or $TS$.

Let $I^*$ be the subsemigroup $\{S, T, ST, TS\}$ of $Q^*$.

**Corollary.** $Q^*$ is the annexed product of $I^*$ and $\{R\}$.

$I^*$ is isomorphic to No. 73 of p. 25 of [4] or No. 100, p. 27 of [5].

The greatest semilattice-decomposition of $I^*$ is $\{S\}, \{T\}, \{ST, TS\}$.

Each congruence class is a null subsemigroup of $I^*$, [6].

The greatest semilattice-decomposition of $Q^*$ is

$$\{S\}, \{T\}, \{R\}, \{RS\}, \{RT\}, \{ST, TS\}, \{RST, RTS\}.$$

Each congruence class is a null subsemigroup of $Q^*$. The greatest factor semilattice of $Q^*$ is isomorphic to the semilattice of all non-empty subsets of a set of three elements with respect to inclusion; and the greatest factor semilattice of $Q^*$ is isomorphic to the annexed product of $\{R\}$ and the greatest factor semilattice of $I^*$.

2. *Congruence-Semigroup.* Let $N^*$ be the semigroup generated by $R, S, T, C$ with (1.1) and

(2.1) $$C^2=C,\ RC=CR,\ SC=CS,\ CTC=TCT=CT$$

**Theorem 2.1.** *$N^*$ is composed of twenty-five elements.*

(2.2) $$S, T, C, SC, ST, TS, CT, TC, STC, CTS, TCS, SCT$$

(2.3) $$R, RS, RT, RC, RSC, RST, RTS, RCT,$$
$$RTC, RSTC, RCTS, RTCS, RSCT.$$

**Proof.** Same as Theorem 1.1, using induction on word length.

Let $U^*$ be the subsemigroup of twelve elements (2.2).

**Corollary.** *$N^*$ is the annexed product of $U^*$ and $\{R\}$.*

The greatest semilattice-decomposition of $N^*$ is

$$\{S\}, \{T\}, \{C\}, \{SC\}, \{ST, TS\}, \{CT, TC\}, \{STC, SCT, CTS, TCS\},$$
$$\{R\}, \{RS\}, \{RT\}, \{RC\}, \{RSC\}, \{RST, RTS\}, \{RCT, RTC\},$$
$$\{RSCT, RSTC, RCTS, RTCS\}.$$

Each congruence class is a null subsemigroup of $N^*$.

3. *Compatible Semigroup.* Let $L^*$ be generated by $R, S, C_r, C_l, T$ with defining relations (1.1) and

(3.1) $$C_rC_l=C_lC_r,\ C_r^2=C_r,\ C_l^2=C_l,\ RC_r=C_rR,\ RC_l=C_lR,\ SC_r=C_rS,$$
$$SC_l=C_lS,\ C_rTC_r=TC_rT=C_rT,\ C_lTC_l=TC_lT=C_lT.$$

**Theorem 3.1.** *$L^*$ is a semigroup consisting of sixty-nine elements. $L^*$ is the annexed product of $W^*$ and $\{R\}$ where $W^*$ is the subsemigroup of $L^*$ generated by $S, C_r, C_l, T$. $W^*$ consists of thirty-four elements:*

$$S, C_r, C_l, T, SC_r, SC_l, ST, TS, C_rC_l, C_rT, TC_r, C_lT,$$
$$TC_l, SC_rC_l, SC_rT,$$
$$STC_r, TSC_r, C_rTS, SC_lT, STC_l, TSC_l, C_lTS, C_rC_lT, C_rTC_l,$$
$$C_lTC_r, TC_rC_l, SC_rC_lT,$$
$$SC_rTC_l, C_rC_lTS, SC_lTC_r, STC_rC_l, C_rTSC_l, C_lTSC_r, TSC_rC_l.$$

**Proof.** As Theorem 1.1.

4. *The Partial Ordering of $N^*$.* From Lemma 2.1 of [2] we learn that the set of all semiclosure operations on a set $E$ form a partially ordered semigroup. Multiplication is by composition of

operations, one applied after another, and the partial ordering is accomplished by $\subseteq$. $P \leq Q$ iff $\rho P \subseteq \rho Q$ for all relations $\rho$ on $E$.

$N^*$ is ordered by saying $P<Q$ (strict), $P \neq Q$ in $N^*$, whenever $P \leq Q$ where now $P$ and $Q$ are considered as semiclosure operators on a set $E$ having more than two elements.

$N^*$ is naturally ordered in the sense that if $A, B \in N^*$, and $A \geq B$, then there is $C \in N^*$ such that $A=CB$ or $A=BC$.

**Theorem 4.1.** *$N^*$ is a semilattice ordered semigroup and satisfies the distributive laws:*

$$(X \vee Y)Z = XZ \vee YZ \text{ and } Z(X \vee Y) = ZX \vee ZY$$

*where $\vee$ means least upper bound.*

We consider first $U^*$. The diagram in [2] shows that $U^*$ is a semilattice with respect to the partial ordering.

To show the distributive laws we consider only the cases where $X$ and $Y$ are incomparable. The cases are:

$S \vee T$, $S \vee C$, $S \vee TC$, $S \vee CT$, $T \vee C$, $T \vee SC$, $C \vee TS$, $C \vee ST$, $TS \vee SC$, $TS \vee TC$,
$TS \vee CT$, $SC \vee TC$, $SC \vee CT$, $ST \vee TCS$, $ST \vee CT$, $ST \vee CTS$, $TCS \vee CT$, $STC \vee CTS$.

The distributive laws in these cases are done by direct computation. One also verifies by computation that

$$XR \vee YR = (X \vee Y)R \text{ and } X \vee YR = (X \vee Y)R$$

hold for $X$ and $Y$ in $U^*$.

Using these results with the fact that $R$ commutes we prove $(X \vee Y)Z = XZ \vee YZ$ and $Z(X \vee Y) = ZX \vee ZY$ for all $X, Y, Z \in N^*$.

**Corollary.** *$Q^*$ is a semilattice ordered semigroup and satisfies the distributive laws.*

**Proof.** $Q^*$ is a subsemigroup of $N^*$ and also a subsemilattice with respect to the partial ordering on $N^*$.

From [2] $RST$ is the greatest element of $Q^*$ and $RSCT$ is the greatest element of $N^*$.

5. *The Equivalence and Congruence Operations Semigroup.* Let $E$ be a set with $\xi$ elements. Let $S_E$ be the set of all relations on $E$. Let $R$, $S$, $T$, and $C$ be operations on $S_E$ defined respectively by (4.1), (4.2), (4.5), and (4.13) of [2]. $Q_\xi$, a homomorphic image of $Q^*$, is the semigroup generated by $R$, $S$, and $T$. Similarly, $N_\xi$ is the semigroup generated by $R$, $S$, $T$, and $C$. If for any $X$, $Y$ in $Q_\xi$ $X \neq Y$ in $Q^*$, one can find a $\rho \in S_E$ and $\rho X \neq \rho Y$, then we say $Q_\xi$ is distinct on $S_E$. If there is a $\rho \in S_E$ such that for all pairs $X$, $Y$ in $Q_\xi$ $X \neq Y$ in $Q^*$, and $\rho X \neq \rho Y$, then we say $Q_\xi$ is strongly distinct on $S_E$. Similar definitions apply to $N_\xi$.

Now, as a trivial case where $|E|=1$, $N_1$ consists of one element, since $R=S=T=C$. Also, $Q_1$ has only one element.

**Theorem 5.2.** *$Q_2$ consists of six elements $R$, $S$, $T$, $RS$, $ST$, $TS$. In this case $RS=RTS=RST$ and $R=RT$. All the elements, however, are not strongly distinct.*

**Corollary.** *$N_2$ is not distinct on $S_E$.*

**Theorem 5.3.** *$Q_3$ and $Q_4$ are isomorphic to $Q^*$. All the elements of $Q_3$ and $Q_4$ are distinct.*

**Proof.** Let $E=\{a, b, c\}$ and use $\rho_1=\{(a, b), (b, a)\}$, $\rho_2=\{(a, b), (c, a)\}$, $\rho_3=\{(a, b), (a, c)\}$. If $E_4=\{a, b, c, d\}$, use $\rho_2$ and $\rho_3$ to give distinctness.

**Lemma 1.** *If $\rho$ gives strong distinctness on $Q_n$ then there exists $(a, b), (b, c)\in\rho$ and $a\neq b, a\neq c, b\neq c$.*

**Proof.** If there were no such pair, $\rho R=\rho RT$. We call this pair the transitivity pair.

**Definition.** $\mathrm{Sym}\,\rho=\{X: X\rho t$ or $t\rho X$ for some $t\}$. $|\mathrm{sym}\,\rho|=$the number of elements in $\mathrm{Sym}\,\rho$. $\mathrm{Sym}\,\rho$ can be called the symbol set of $\rho$. $\iota$ is the identity relation on $E$ (the diagonal).

**Lemma 2.** *If $\rho$ gives strong distinctness on $Q_n$ then $|\mathrm{Sym}\,(\rho-\iota)|\neq n$.*

**Proof.** Suppose $|\mathrm{Sym}\,(\rho-\iota)|=n$. Then $\rho ST=\rho RST$.

**Theorem 5.4.** *$Q_3$ is not strongly distinct.*

**Proof.** Use Lemma 1, $|\mathrm{Sym}\,(\rho-\iota)|=3$ and Lemma 2.

**Lemma 3.** *If $|\mathrm{Sym}\,(\rho-\iota)|=3$ and $\rho$ has a transitivity pair then $\rho RST=\rho RTS$.*

**Proof.** Let $\mathrm{Sym}\,(\rho-\iota)=\{a, b, c\}$. Then $\rho RST=\rho RTS=\{(a, b), (b, a), (b, c), (c, b), (a, c), (c, a)\}\cup\iota$.

**Theorem 5.5.** *$Q_4$ is not strongly distinct.*

**Proof.** Suppose $\rho$ gives strong distinctness. Lemma 1 implies $|\mathrm{Sym}\,(\rho-\iota)|\geq 3$. Lemma 2 implies $|\mathrm{Sym}\,(\rho-\iota)|\neq 4$ $\therefore$ $|\mathrm{Sym}\,(\rho-\iota)|=3$. Lemma 3 implies $\rho RST=\rho RTS$.

**Theorem 5.6.** *$Q_5$ is strongly distinct and isomorphic to $Q^*$.*

**Proof.** $\rho=\{(a, b), (b, c), (d, c)\}$ where $E=\{a, b, c, d, e\}$.

**Theorem 5.7.** *$N_3$ and $N_4$ are not strongly distinct. Since $Q_\xi$ is a subsemigroup of $N_\xi$ we have from Theorems 5.4 and 5.5 that $N_3$ and $N_4$ are not strongly distinct.*

**Theorem 5.8.** *$N_5$ is not strongly distinct.*

**Proof.** Assume $\rho$ gives strong distinctness. Lemma 2 and Lemma 3 imply $|\mathrm{Sym}\,(\rho-\iota)|=4$. Assume $\mathrm{Sym}\,(\rho-\iota)=\{a, b, c, d\}$. Also by Lemma 1 $\rho$ has a transitivity pair. We may assume it to be $(a, b)$, $(b, c)$. One of the following sets must be a subset of $\rho$.

1) $\{(a, b), (b, c), (d, a)\}$
2) $\{(a, b), (b, c), (d, b)\}$
3) $\{(a, b), (b, c), (d, c)\}$
4) $\{(a, b), (b, c), (a, d)\}$
5) $\{(a, b), (b, c), (b, d)\}$
6) $\{(a, b), (b, c), (c, d)\}$

One can verify that $|\rho RST|=17$. If $\omega$ is the universal relation, let

Sym $\omega=\{a, b, c, d, e\}$, $|\omega|=25$ $\therefore$ $|\omega-\rho RST|=8$. $\omega-\rho RST$ contains all off diagonal pairs which have an $e$, for example $(e, a)$ or $(b, e)$. If $\rho C\cap(\omega-\rho RST)=\phi$ then $\rho CRST=\rho RST$; if $\rho C\cap(\omega-\rho RST)\neq\phi$ then $|\mathrm{Sym}(\rho C-\iota)|=5$. But then by Lemma 2, $(\rho C)ST=(\rho C)RST$ or $\rho CST=\rho CRST$.

**Theorem 5.9.** *$N_3$ is distinct.*

**Proof.** This result was obtained by a computer (*CDC* 3600) calculation. The semigroup which may be used is given by:

| | | |
|---|---|---|
| $a$ | $a$ | $a$ |
| $a$ | $a$ | $a$ |
| $a$ | $a$ | $c$ |

Not all semigroups of order three gave distinctness. What can be said about semigroups which collapse $N_\xi$? For every $\xi$ there exists at least one semigroup which collapses $N_\xi$, namely the null semigroup.

**Theorem 5.10.** *$N_4$ is distinct.*

**Proof.** Adjoin an identity to the semigroup used in $N_3$.

**Theorem 5.11.** *$N_\xi$ for $\xi\geq 6$ is strongly distinct.*

**Proof.** A semigroup $G$ is defined as follows: $G=\{a, b, c, d, e\}\cup F$, where $F$ is arbitrary and non-empty and $\{a, b, c, d, e\}\cap F=\phi$. $b^2=e$, $xy=a$ if $x\neq b$ and $y\neq b$. Let $\rho=\{(a, b), (b, c), (d, c)\}$.

**Corollary.** *$Q_\xi$ for $\xi\geq 6$ is strongly distinct.*

**Proof.** $Q_\xi$ is a subsemigroup of $N_\xi$.

**Note.** $L^*$ is partially ordered in the same sense as $N^*$.

**Unsolved problem.** Does there exist a semigroup such that $L=L^*$?

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