9. An Algebraic Formulation of K-N Propositional Calculus. II

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In his paper [1], K. Iséki has defined K-N algebra as follows: Let X be an abstract algebra consisting of $0, p, q, \cdots$, with a binary operation $*$ and a unary operation $\sim$ satisfying the following conditions:

a) $\sim(p*p)*p=0$,

b) $\sim p*(q*p)=0$,

c) $\sim\sim(\sim\sim(p*r)*\sim(r*q))*\sim(\sim q*p)=0$,

d) $\sim\sim\beta*\sim\alpha=0$ and $\alpha=0$ imply $\beta=0$, where α, β are expressions in X.

In this paper, we shall show that the NK-algebra is characterized by the following conditions:

1) $\sim(p*p)*p=0$,

2) $\sim q*(q*p)=0$,

3) $\sim\sim(\sim\sim(p*r)*\sim(r*q))*\sim(\sim q*p)=0$,

4) $\sim\sim\beta*\sim\alpha=0$ and $\alpha=0$ imply $\beta=0$, where α, β are expressions (For the details on N-K propositional calculus, see [2], [3], [4].)

K. Iséki has proved that the NK-algebra implies $\sim q*(q*p)=0$. Therefore we shall prove that 1), 2), 3), and 4) imply b).

A) $\sim\alpha*\beta=0$ implies $\sim\sim(\beta*\gamma)*\sim(\gamma*\alpha)=0$.

Proof. In 3), put $p=\beta$, $q=\alpha$, $r=\gamma$, then by 4) we have A). Then we have

B) $\sim\alpha*\beta=0$, $\sim\gamma*\alpha=0$ imply $\beta*\sim\gamma=0$.

In A), put $\alpha=p*p$, $\beta=p$, $\gamma=\sim p$, then $\sim(p*p)*p=0$ implies $\sim\sim(p*\sim p)*\sim(\sim p*(p*p))=0$.

By 2), we have

5) $p*\sim p=0$.

In 3), put $p=\sim\sim q$, $r=\sim r$, then

$$\sim\sim(\sim\sim(\sim\sim q*\sim r)*\sim(\sim r*q))*\sim(\sim q*\sim\sim q)=0.$$

And In 3), put $p=\sim\sim q$, then

$$\sim\sim(\sim\sim(\sim\sim q*r)*\sim(r*q))*\sim(\sim q*\sim\sim q)=0.$$

By 5), $\sim q*\sim\sim q=0$, hence we have

6_1) $\sim\sim(\sim\sim q*\sim r)*\sim(\sim r*q)=0$,

and

6_2) $\sim\sim(\sim\sim q*r)*\sim(r*q)=0$.

These expressions mean

C) $\alpha*\beta=0$ implies $\sim\sim\beta*\alpha=0$ and $\sim\sim\alpha*\sim\sim\beta=0$.

In 6_2), put $r=p, q=\sim\sim p$, then

$$\sim\sim(\sim\sim\sim p*p)*\sim(p*\sim p)=0.$$

By 5), we have

7) $\sim\sim\sim p*p=0$.

In 3), put $p=\sim\beta, q=\sim\alpha, r=\alpha$,

$$\sim\sim(\sim\sim(\sim\beta*\alpha)*\sim(\alpha*\sim\alpha))*\sim(\sim\sim\alpha*\sim\beta)=0.$$

By 5), $\alpha*\sim\alpha=0$, hence we have

D) $\sim\sim\alpha*\sim\beta=0$ implies $\sim\beta*\alpha=0$.

In 3), put $p=\alpha, q=\beta, r=\gamma$, then

$$\sim\sim(\sim\sim(\alpha*\gamma)*(\gamma*\beta))*\sim(\sim\beta*\alpha)=0.$$

And by D) $\sim\sim(\alpha*\gamma)*\sim(\gamma*\beta)=0$ implies $\sim(\gamma*\beta)*(\alpha*\gamma)=0$. Therefore, let $\sim\beta*\alpha=0$, then we have

E) $\sim\beta*\alpha=0$ implies $\sim(\gamma*\beta)*(\alpha*\gamma)=0$.

From E), we have the following variations:

$$\sim\alpha*\beta=0 \quad \text{implies} \quad \sim(\delta*\alpha)*(\beta*\delta)=0,$$
$$\sim\gamma*\delta=0 \quad \text{implies} \quad \sim(\alpha*\gamma)*(\delta*\alpha)=0.$$

By B) and the above variations, we have

F) $\sim\alpha*\beta=0, \sim\gamma*\delta=0$ imply $(\beta*\delta)*\sim(\alpha*\gamma)=0$.

In E), put $\alpha=\sim\sim p, \beta=p, \gamma=r$, then

$$\sim p*\sim\sim p \quad \text{implies} \quad \sim(r*p)*(\sim\sim p*r)=0.$$

By 5), $\sim p*\sim\sim p=0$, hence we have

8) $\sim(r*p)*(\sim\sim p*r)=0$.

In 7), $p=\sim\alpha$, then $\sim\sim\sim\sim\alpha*\sim\alpha=0$. Therefore, let $\alpha=0$, then we have $\sim\sim=0$, that is,

G) $\alpha=0$ implies $\sim\sim\alpha=0$.

In G), put $\alpha=\sim\gamma*\beta$, then we have

(1) $\sim\gamma*\beta=0$ implies $\sim\sim(\sim\gamma*\beta)=0$.

In 6_1), put $r=\delta, q=\gamma$, then we have

(2) $\sim\sim(\sim\sim\gamma*\sim\delta)*\sim(\sim\delta*\gamma)=0$ implies $\sim\sim\gamma*\sim\delta=0$.

In F), put $\alpha=\sim\gamma, \beta=\sim\delta, \gamma=\beta, \delta=\alpha$, then

(3) $\sim\sim\gamma*\sim\delta=0, \sim\beta*\alpha=0$ imply $(\sim\delta*\alpha)*\sim(\sim\gamma*\beta)=0$.

In C), put $\alpha=\sim\delta*\alpha, \beta=\sim(\sim\gamma*\beta)$, then we have

(4) $(\sim\delta*\alpha)*\sim(\sim\gamma*\beta)=0$ implies

$$\sim\sim(\sim\delta*\alpha)*\sim\sim\sim(\sim\gamma*\beta)=0.$$

From (4), if we let $\sim\sim(\sim\gamma*\beta)=0$, then by 4), we have $\sim\delta*\alpha=0$. Therefore, from (1), (2), (3), and (4) we have

H) $\sim\beta*\alpha=0, \sim\gamma*\beta=0, \sim\delta*\gamma=0$ imply $\sim\delta*\alpha=0$.

Put $p=\sim\sim p$ in 1), $r=\sim\sim p$, and $r=p$ in 8), then we have respectively

$$\sim(\sim\sim p*\sim\sim p)*\sim\sim p=0,$$
$$\sim(\sim\sim p*p)*(\sim\sim p*\sim\sim p)=0,$$

$$\sim(p*p)*(\sim\sim p*p)=0.$$

By H), we have $\sim(p*p)*\sim\sim p=0$.

On the other hand, putting $q=p$ in 2), we have $\sim p*(p*p)=0$. By B), we have $\sim\sim p*\sim p=0$, further by D) we have

9) $\sim p*p=0$.

In E), put $\beta=p, \alpha=p, \gamma=r$, then by 9), we have

10) $\sim(r*p)*(p*r)=0$.

In H), put $\delta=\gamma$, then by 9) we have

I) $\sim\beta*\alpha=0, \sim\gamma*\beta=0$ imply $\sim\gamma*\alpha=0$.

Put $r=p, p=q$ in 10) and $q=p, p=q$ in 2), then we have

$$\sim(p*q)*(q*p)=0, \sim p*(p*q)=0.$$

Hence by I), we have

11) $\sim p*(q*p)=0$.

Therefore the proof is complete.

References

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