6. *On an Equivalence of Convergences in Ranked Spaces*

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K. Kunugi [1] has given the definitions of ranked spaces and convergences in ranked spaces.

One of convergences is defined as follows:

Let α be a natural number. Let $\{p_\alpha\}$ be a sequence of points and p a point in the ranked spaces R. Suppose that there is a decreasing sequence of the neighborhoods $V_\alpha(p)$ of the point p that has the rank γ_α, and that each of the neighborhoods $V_\alpha(p)$ satisfies the following conditions:

1) $V_1(p) \supseteq V_2(p) \supseteq \cdots \supseteq V_\alpha(p) \supseteq \cdots$,
2) $V_\alpha(p) \in \mathfrak{U}_{\gamma_\alpha}$, where $V_\alpha(p) \in \mathfrak{U}_{\gamma_\alpha}$ means that γ_α is the rank of $V_\alpha(p)$,
3) $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_\alpha \leq \cdots$,
4) $p_\alpha \in V_\alpha(p)$ for all α,
5) $\lim_{\alpha\to\infty} \gamma_\alpha = +\infty$.

Then we say that the sequence $\{p_\alpha\}$ converges to the point p, and we write the following:

$$p \in \left\{\lim_{\alpha} p_\alpha\right\}.$$

In this paper we shall define the para convergence and consider the relation between the convergence and the para convergence. To distinguish between the former and the latter, we shall say that a sequence $\{p_\alpha\}$ is R-convergent to the point p, if the sequence $\{p_\alpha\}$ converges to the point p.

Definition of the para convergence. Let α be a natural number. Let $\{p_\alpha\}$ be a sequence of points and p a point in the ranked space R. Suppose that there is a decreasing sequence of the neighbordoods $V_\alpha(p_\alpha)$ of p_α that has the rank γ_α, and that each of the neighborhoods $V_\alpha(p_\alpha)$ satisfies the following conditions,

1′) $V_1(p_1) \supseteq V_2(p_2) \supseteq \cdots \supseteq V_\alpha(p_\alpha) \supseteq \cdots$,
2′) $V_\alpha(p_\alpha) \in \mathfrak{U}_{\gamma_\alpha}$,
3′) $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_\alpha \leq \cdots$,
4′) $p \in V_\alpha(p_\alpha)$ for all α,
5′) $\lim_{\alpha\to\infty} \gamma_\alpha = +\infty$.

Then we say that the sequence $\{p_\alpha\}$ is para convergent to the point

p, and we write the following:

$$p \in \left\{ \text{para} \lim_{\alpha} p_\alpha \right\}.$$

Proposition 1. Let us suppose that the ranked space satisfies the two following conditions.

(1) If $V_\alpha(p)$ is the neighborhood of the point p, and if $V_\alpha(p) \in \mathfrak{A}_{\gamma_\alpha}$, $V_\alpha(p) \supseteq V(q)$ and $x \in V_\alpha(p)$, then there is a neighborhood $U_{\alpha_k}(x)$ of the point x such that $V(q) \subseteq U_{\alpha_k}(x) \in \mathfrak{A}_{\left[\frac{\gamma_\alpha}{k}\right]}$, where k is a natural number determined by the space and $\left[\frac{\gamma_\alpha}{k}\right]$ is a Gaussian symbol.

(2) If $V_\alpha(p) \in \mathfrak{A}_{\gamma_\alpha}$, $V_\beta(q) \in \mathfrak{A}_{\gamma_\beta}$, and $p \in V_\alpha(p) \cap V_\beta(q)$, then $V_\alpha(p) \cap V_\beta(q)$ is the neighborhood of the point p that has the rank γ_α.

Then the sequence $\{p_\alpha\}$ is R-convergent to the point p, if and only if the sequence $\{p_\alpha\}$ is para convergent to the point p.

Proof. Let the sequence $\{p_\alpha\}$ be R-convergent to the point p. We have the sequence of the neighborhoods $V_\alpha(p)$ where 1), 2), 3), 4), and 5) are contented. Hence $V_\alpha(p) \in \mathfrak{A}_{\gamma_\alpha}$, $V_{\alpha+1}(p) \subseteq V_\alpha(p)$, and $p \in V_\alpha(p)$. Therefore by the condition (1), there is the neighborhood $U_{\alpha_k}(p_\alpha)$ of the point p_α that has the rank $\left[\frac{\gamma_\alpha}{k}\right]$, which satisfies

$$V_{\alpha+1}(p) \subseteq U_{\alpha_k}(p_\alpha). \quad \cdots ①$$

While as is easily seen, $U_{\alpha_k}(p_\alpha) \in \mathfrak{A}_{\left[\frac{\gamma_\alpha}{k}\right]}$, $V_\alpha(p) \in \mathfrak{A}_{\gamma_\alpha}$, and $p_\alpha \in U_{\alpha_k}(p_\alpha) \cap V_\alpha(p)$. By the condition (2), $U_{\alpha_k}(p_\alpha) \cap V_\alpha(p) \equiv W_{\alpha_k}(p_\alpha)$ is the neighborhood of the point p_α that has the rank $\left[\frac{\gamma_\alpha}{k}\right]$. Similarly, $W_{(\alpha+1)_k}(p_{\alpha+1})$ is the neighborhood of the point $p_{\alpha+1}$ that has the rank $\left[\frac{\gamma_{\alpha+1}}{k}\right]$. We have

$$W_{(\alpha+1)_k}(p_{\alpha+1}) \subseteq W_{\alpha_k}(p_\alpha),$$

because, if $x \in W_{(\alpha+1)_k}(p_{\alpha+1})$ then $x \in V_{\alpha+1}(p)$, and $x \in V_\alpha(p)$, and using ① and $x \in U_{\alpha_k}(p_\alpha)$, $x \in W_{\alpha_k}(p_\alpha)$ follows. Therefore there exists the decreasing sequence of the neighborhoods $W_{\alpha_k}(p_\alpha)$ of the point p_α that has the rank $\left[\frac{\gamma_\alpha}{k}\right]$, and $\lim_{\alpha\to\infty}\left[\frac{\gamma_\alpha}{k}\right] = +\infty$ since $\lim_{\alpha\to\infty} \gamma_\alpha = +\infty$. Namely,

$$W_{1_k}(p_1) \supseteq W_{2_k}(p_2) \supseteq \cdots \supseteq W_{\alpha_k}(p_\alpha) \supseteq \cdots.$$

By ① $p \in U_{\alpha_k}(p_\alpha)$, and hence $p \in W_{\alpha_k}(p_\alpha)$. Thus we have

$$p \in \left\{ \text{para} \lim_{\alpha} p_\alpha \right\}.$$

Similarly, we can show that the converse is true.

Proposition 2. Let us suppose that a sequence $\{p_\alpha\}$ is R-convergent to the point p. Then the sequence $\{p_\alpha\}$ is para convergent to the point p, if and only if the following conditions are satisfied:

a) For every neighborhood $V_\alpha(p)$ of the point p that appears in 1), there exists $U_\alpha(P_\alpha)$ such that

$$p_\alpha,\ p \in U_\alpha(p_\alpha),\ U_\alpha(p_\alpha) \in \mathfrak{A}_{\gamma'_\alpha},$$

where γ'_α is a natural number determined by p and p_α,

b) $$U_\alpha(p_\alpha) \supseteq U_{\alpha+1}(p_{\alpha+1}),$$
$$\gamma'_\alpha \leq \gamma'_{\alpha+1},$$

c) $$\lim_{\alpha\to\infty} \gamma'_\alpha = +\infty.$$

Proof. Let $\{p_\alpha\}$ be R-convergent to the point p and $\{p_\alpha\}$ be para convergent to the same point p, then there exists the neighborhood $U_\alpha(p_\alpha)$ corresponding to $V_\alpha(p)$ such that

$$U_1(p_1) \supseteq U_2(p_2) \supseteq \cdots \supseteq U_\alpha(p_\alpha) \supseteq \cdots,$$
$$\gamma'_1 \leq \gamma'_2 \leq \cdots \leq \gamma'_\alpha \leq \cdots, \qquad \lim_{\alpha\to\infty} \gamma'_\alpha = +\infty.$$

Hence it is clear that the three conditions a), b), and c) are contented.

On the other hand, let $\{p_\alpha\}$ be R-convergent to the point p and let the three conditions a), b), and c) be contented. We can define the neighborhood $U_\alpha(p_\alpha) \in \mathfrak{A}_{\gamma'_\alpha}$ by the condition a). Therefore using the conditions b) and c), we can see that $\{p_\alpha\}$ is para convergent to the point p.

Proposition 2′. Let us suppose that the sequence $\{p_\alpha\}$ is para convergent to the point p. Then the sequence $\{p_\alpha\}$ is R-convergent to the point p if and only if the following conditions are satisfied:

a′) For every neighborhood $U_\alpha(p_\alpha)$ that appears in 1′), there exists $V_\alpha(p)$ such that

$$p_\alpha,\ p \in V_\alpha(p),\ V_\alpha(p) \in \mathfrak{A}_{\gamma'_\alpha},$$

where γ'_α is a natural number determined by p and p_α,

b′) $$V_\alpha(p) \supseteq V_{\alpha+1}(p),$$
$$\gamma'_\alpha \leq \gamma'_{\alpha+1},$$

c′) $$\lim_{\alpha\to\infty} \gamma'_\alpha = +\infty.$$

Proof. We can prove this proposition by the same procedure as in the former proposition.

Proposition 3. Let the following condition (A) be satisfied in the ranked space.

(A) If $V(p) \in \mathfrak{A}_\gamma$ and $x \in V(p)$, then $V(p)(\equiv U(x))$ is the neighborhood of the point x that has the rank $\left[\frac{\gamma}{k}\right]$, where k is a natural number fixed by the space and $\left[\frac{\gamma}{k}\right]$ is a Gaussian symbol.

Then the sequence $\{p_\alpha\}$ of points is R-convergent to the point p if and only if the sequence $\{p_\alpha\}$ of points is para convergent to the point p.

Proof. Let $\{p_\alpha\}$ be R-convergent to the point p. We have the sequence of the neighborhoods $V_\alpha(p)$ such that 1), 2), 3), 4), and 5) are contented. Now, we shall verify that conditions a), b), and c) in the proposition 2 are satisfied.

a) Since $p, p_\alpha \in V_\alpha(p)$ and by the condition (A) there exists the neighborhood $U_\alpha(p_\alpha)$ such that $U_\alpha(p_\alpha)(\equiv V_\alpha(p)) \in \mathfrak{A}_{\left[\frac{\gamma_\alpha}{k}\right]}$.

b) If $V_\alpha(p) \supseteq V_{\alpha+1}(p)$, then it is clear that

$$U_\alpha(p_\alpha) \equiv V_\alpha(p) \supseteq V_{\alpha+1}(p) \equiv U_{\alpha+1}(p_{\alpha+1})$$

and $\left[\frac{\gamma_\alpha}{k}\right] \leq \left[\frac{\gamma_{\alpha+1}}{k}\right]$.

c) $\lim_{\alpha\to\infty}\left[\frac{\gamma_\alpha}{k}\right] = +\infty$ is obvious.

Hence the sequence $\{p_\alpha\}$ is para convergent to the point p by the proposition 2.

Conversely, if $\{p_\alpha\}$ is para convergent to the point p, then by the condition (A) we can verify that the three conditions a), b), and c) in the proposition 2 are satisfied. Hence $\{p_\alpha\}$ is R-convergent to the point p.

Example. Let R be a metric space and the neighborhood of the point p be a subset containing the point p. We define the rank of the neighborhood $V(p)$ as follows,

if $V(p) \subseteq \left\{x \,\middle|\, \rho(x, p) \leq \frac{1}{n}\right\}$ then $V(p)$ has the rank n, and

if $V(p) \supset \{x \mid \rho(x, p) > 1\}$ then $V(p)$ has the rank 0.

In this case, the metric space R is considered as a ranked space. We can verify that the condition (A) in the proposition 3 is satisfied. To verify that the condition (A) is satisfied, we may take 2 as the natural number k in the condition (A). In fact, if $V(p) \in \mathfrak{A}_\gamma$ and $x \in V(p)$, then $V(p) \equiv U(x) \in \mathfrak{A}_{\left[\frac{\gamma}{2}\right]}$, because, as the figure shows, $V(p)$ is contained in the sphere with the center x and the radius $1\Big/\left[\frac{\gamma}{2}\right]$. Hence the R-convergence is equivalent to the para convergence in this space.

Moreover the conditions (1) and (2) in the proposition 1 held in this ranked space.

Remark 1. The condition (A) in the proposition 3 follows from the two conditions (1) and (2) in the proposition 1.

In fact, we consider $V(q)$ of the condition (1) in the proposition 1 as $V_\alpha(p)$ itself, then $V_\alpha(p) \subseteq U_{\alpha_k}(x) \in \mathfrak{A}_{\left[\frac{\gamma_\alpha}{k}\right]}$. Hence

$$x \in V_\alpha(p) \cap U_{\alpha_k}(x) = V_\alpha(p).$$

Therefore $V_\alpha(p)$ is the neighborhood of the point x that has the rank

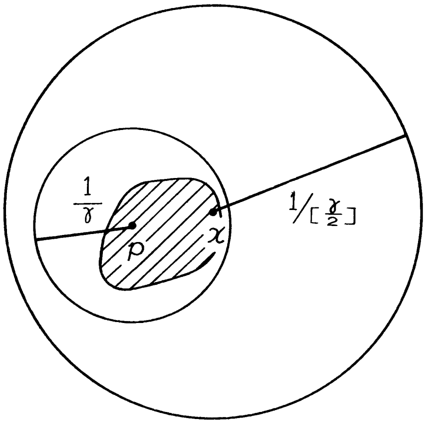

Fig. 1

$\left[\frac{\gamma_\alpha}{k}\right]$ by the conditions in the proposition 1.

Remark 2. An other proof of the proposition 1 follows from the remark 1.

Remark 3. The condition (1) in the proposition 1 follows from the condition (A) in the proposition 3.

In fact, $V_\alpha(p)$ of the condition (1) in the proposition 1 is the neighborhood $U_{\alpha_k}(x)$ of the point x that has the rank $\left[\frac{\gamma_\alpha}{k}\right]$ by the condition (A), hence

$$V(q) \subseteq V_\alpha(p) = U_{\alpha_k}(x) \in \mathfrak{A}_{\left[\frac{\gamma_\alpha}{k}\right]}.$$

Remark 4. The condition (2) in the proposition 1 dose not follow from the condition (A) in the proposition 3.

Example. Let R be a metric space and the neighborhood of the point p in R be any sphere in R that contains the point p.

We define the rank of the neighborhood $V(p)$ as the inverse number of the radius of the sphere.

In this case, the metric space R is considered as a ranked space and the condition (A) in the proposition 3 is satisfied.

However, if the two neighborhoods $V(p)$ and $U(p)$ of the point p have a non-empty intersection, then $V(p) \cap U(p)$ is not the neighborhood of the point p, because $V(p) \cap U(p)$ is not a sphere.

Reference

[1] K. Kunugi: Sur la méthode des espaces rangés. I. Proc. Japan Acad., **42** (4), 318-322 (1966).