19. Concrete Characterization of the Domains of Fractional Powers of Some Elliptic Differential Operators of the Second Order

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1. **Introduction.** Recently J. L. Lions has succeeded in characterizing the domains of the fractional powers of an arbitrary regularly accretive operator[1] A in a Hilbert space H as interpolation spaces between the domain of the operator and the whole Hilbert space H.[2] In this connection it would be worth while to characterize domains of fractional powers of differential operators as concretely as possible. However it seems that little is known in this direction.

In the present note we shall be concerned with the following problem: Let Ω be a bounded domain in $\boldsymbol{R}^m$ with the boundary $\partial\Omega$ which is a $m-1$ dimensional C^∞ manifold. The unit outer normal to $\partial\Omega$ is denoted by n. Let A_α be a regularly accretive operator in $L^2(\Omega)$ mapping $D_\alpha=\left\{u \in H^2(\Omega);\ \alpha\frac{\partial u}{\partial n}+(1-\alpha)u|_{\partial\Omega}=0\right\}$ into $L^2(\Omega)$, where $H^s(\Omega)$, $s\geq 0$, denotes the Sobolev space of order s and where $\alpha\equiv\alpha(x')$ is a given C^∞ function defined for x' in $\partial\Omega$ with $0\leq\alpha\leq 1$. Then, we ask how one can characterize the domain $D(A_\alpha^{1-\theta})$, $0\leq\theta\leq 1$, of the fractional power $A_\alpha^{1-\theta}$ of A_α. The purpose of the present note is to give a fairly satisfactory concrete answer to this question assuming that on each of connected components of $\partial\Omega$ either α vanishes identically or α is never equal to zero.[3]

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§2. **Results.** Since the boundary $\partial\Omega$ is smooth, there is a system of first order differential operators $\tilde{D}_1, \cdots, \tilde{D}_{m-1}$, the restriction of which to the boundary forms a basis of first order differential operators tangential to $\partial\Omega$. $\zeta(x)$ denotes the distance from x in $\boldsymbol{R}^m$ to $\partial\Omega$.

Now we state the result in the case of $\alpha\equiv 0$, that is, the Dirichlet boundary condition.

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1) See Definition 2.1 and Theorem 2.1 in [1].

2) See Théorème 3.1 in [2].

3) Detailed proofs of theorems in this note will be published elsewhere.

Theorem 1.[4] If α vanishes identically on $\partial\Omega$, then the domain $D(A_0^{1-\theta})$ of the fractional power $A_0^{1-\theta}$, $0\leq\theta\leq 1$, of A_0 is given by

(a) $D(A_0^{1-\theta})=E^{0,2(1-\theta)}(\Omega)=H^{2(1-\theta)}(\Omega)$, $\qquad 0\leq 1-\theta\leq\frac{1}{4}$,

(b) $D(A_0^{1/4})=E^{0,1/2}(\Omega)\subset H^{1/2}(\Omega)$,

(c) $D(A_0^{1-\theta})=E^{0,2(1-\theta)}(\Omega)=H_{\gamma_0}^{2(1-\theta)}(\Omega)$, $\qquad \frac{1}{4}<1-\theta<\frac{1}{2}$,

(d) $D(A_0^{1/2})=E^{1,0}(\Omega)=H_{\gamma_0}^{1}(\Omega)$,

(e) $D(A_0^{1-\theta})=E^{1,2(1-\theta)-1}(\Omega)=H_{\gamma_0}^{2(1-\theta)}(\Omega)$, $\qquad \frac{1}{2}<1-\theta<\frac{3}{4}$,

(f) $D(A_0^{3/4})=E^{1,1/2}(\Omega)\subset H_{\gamma_0}^{3/2}(\Omega)$,

(g) $D(A_0^{1-\theta})=E^{1,2(1-\theta)-1}=H_{\gamma_0}^{2(1-\theta)}(\Omega)$, $\qquad \frac{3}{4}<1-\theta<1$,

(h) $D(A_0)=H_{\gamma_0}^{2}(\Omega)$,

where, $E^{0,s}(\Omega)=\left\{u\in H^s(\Omega);\ \tilde{K}_s^2(u)\equiv\int_\Omega \zeta(x)^{-2s}\,|\,u(x)\,|^2\,dx<\infty\right\}$,

$$E^{1,s}(\Omega)=\left\{u\in H_{\gamma_0}^{1+s}(\Omega);\ \sum_{j=1}^{m-1}\tilde{K}_s^2(\tilde{D}_j u)<\infty\right\},\qquad 0\leq s<1$$

and where $H_{\gamma_0}^s(\Omega)=\{u\in H^s(\Omega);\ u\,|_{\partial\Omega}=0\}$, $\qquad s>\frac{1}{2}$.

In the case of non zero α, we have the following Theorem 2, where we have extended the function α over to $\bar{\Omega}=\Omega+\partial\Omega$ appropriately.

Theorem 2. If α never vanishes on the boundary $\partial\Omega$, then the domain $D(A_\alpha^{1-\theta})$ of the fractional power $A_\alpha^{1-\theta}$, $0\leq\theta\leq 1$, of the operator A_α, is given by

(a) $D(A_\alpha^{1-\theta})=H^{2(1-\theta)}(\Omega)$, $\qquad 0\leq 1-\theta<\frac{3}{4}$,[5]

(b) $D(A_\alpha^{3/4})=M_\alpha^{3/2}(\Omega)\subset H^{3/2}(\Omega)$,

(c) $D(A_\alpha^{1-\theta})=M_\alpha^{2(1-\theta)}(\Omega)=H_\alpha^{2(1-\theta)}(\Omega)$, $\qquad \frac{3}{4}<1-\theta<1$,

(d) $D(A_\alpha)=H_\alpha^2(\Omega)$,

where

$$M_\alpha^{1+s}(\Omega)=\left\{u\in H^{1+s}(\Omega);\ \int_\Omega \zeta(x)^{-2s}\left|\left(\alpha(x)\frac{\partial u(x)}{\partial\zeta}+(1-\alpha(x))u(x)\right)\right|^2 dx<\infty\right\},$$

$$0<s<1,$$

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4) In the case of $0<1-\theta<\frac{1}{2}$, $1-\theta\neq\frac{1}{4}$, the result was essentially known. See Théorème 5.1 of [3].

5) Therefore $D(A_\alpha^{1-\theta})$ is free from the boundary condition in this case. This was already proved in the case of $0<1-\theta<\frac{1}{4}$, by a more general method. See [2], [4].

and $H^s_\alpha(\Omega)=\left\{u\in H^s(\Omega);\ \alpha\dfrac{\partial u}{\partial n}+(1-\alpha)u\,|_{\partial\Omega}=0\right\}$, $\quad \dfrac{3}{2}<s$.

Since it is rather lengthy, we omit the full statement of our result for the general case, where $\partial\Omega$ consists of more than one connected components and α vanishes identically on some of them and never vanishes on the other. The result for the general case is a simple consequence of Theorems 1, 2, and the following.

Theorem 3. Take any θ in $0\le\theta\le1$, then a function u belongs to $D(A_\alpha^{1-\theta})$ if and only if φu belongs to $D(A_\alpha^{1-\theta})$ for any φ in $C^\infty(\bar\Omega)$ with $\left.\dfrac{\partial\varphi}{\partial n}\right|_{\partial\Omega}=0$.

Corollary. Let G be any relatively compact open subset in Ω and let W^s_G be the space of functions of $H^s(\Omega)$ vanishing outside G. Then $W_G^{2(1-\theta)}$, $0<\theta<1$, is contained in the domain $D(A_\alpha^{1-\theta})$, whatever α may be. Moreover we have, for some constant $c>0$,

$$\|u\|_{D(A_\alpha^{1-\theta})}\le c\,\|u\|_{H^{2(1-\theta)}(\Omega)},\qquad \text{for all } u \text{ in } W_G^{2(1-\theta)}.$$

Here and hereafter C denotes generic constants and $\|u\|_V$ denotes the norm of a function u in a function space V.

§3. Outline of the proofs. We begin with *the proof of Theorem* 3. The if part of the assertion follows from the facts that there exists a partition of unity $\{\varphi_j\}_{j=1}^N$ satisfying $\left|\dfrac{\partial\varphi_j}{\partial n}\right|_{\partial\Omega}=0$, $1\le j\le N$, and that the domain $D(A_\alpha^{1-\theta})$ is a linear space.

Since the multiplication operator $\varphi\,\cdot$ continuously maps $L^2(\Omega)$ into $L^2(\Omega)$ and D_α into D_α respectively, this also maps $D(A_\alpha^{1-\theta})$ into $D(A_\alpha^{1-\theta})$ continuously.[6]

Proof of Theorems 1 *and* 2. Although the half space $\boldsymbol{R}^m_+=\{x=(x',x_m)\,|\,x'\in\boldsymbol{R}^{m-1},\ x_m>0\}$ is unbounded, we shall first prove Theorems 1 and 2 in the case that $\Omega=\boldsymbol{R}^m_+$. Introduce the following maps λ and μ;

$$\lambda: L^2(\boldsymbol{R}^m_+)\to L^2(\boldsymbol{R}^m),\qquad \lambda u(x',x_m)=\begin{cases}u(x',x_m), & x_m>0\\ -u(x',-x_m), & x_m\le0,\end{cases}$$

$$\mu: L^2(\boldsymbol{R}^m)\to L^2(\boldsymbol{R}^m_+),\qquad \mu v(x',x_m)=\frac{1}{2}[v(x',x_m)-v(x',x_m)]|_{\boldsymbol{R}^m_+}.$$

Then λ maps $D_0\equiv D(A_0)$ continuously into $H^2(\boldsymbol{R}^m)$ and μ maps $H^2(\boldsymbol{R}^m)$ into D_0 continuously. Moreover, $\mu\circ\lambda$ is the identity. Interpolating these maps, we have

$$\|\lambda u\|^2_{H^{2(1-\theta)}(\boldsymbol{R}^m)}\le c\,\|u\|^2_{D(A_0^{1-\theta})},\qquad (u\in D(A_0^{1-\theta}),) \tag{1}$$

and

6) Here we have used the fundamental theorem due to J. L. Lions that $D(A_\alpha^{1-\theta})$ is an interpolation space between $D\alpha$ and $L^2(\Omega)$. See Théorème 3.1 in [2].

(2) $$\| \mu v \|^2_{D(A_0^{1-\theta})} \le c \| v \|^2_{H^{2(1-\theta)}(\boldsymbol{R}^m)} \qquad (v \in H^{2(1-\theta)}(\boldsymbol{R}^m)).$$[7]

Replacing v in (2) by λu and using (1), we have

(3) $$c^{-1} \| u \|^2_{D(A_0^{1-\theta})} \le \| \lambda u \|^2_{H^{2(1-\theta)}(\mathbf{R}^m)} \le c \| u \|^2_{D(A_0^{1-\theta})}$$

for all u in $D(A_0^{1-\theta})$.

Recall that for any u in $H^s(\boldsymbol{R}^m)$, $s \ge 0$,

(4) $$\begin{cases} \| v \|^2_{H^s(\boldsymbol{R}^m)} = \sum_{|\alpha| \le s} \| D^\alpha v \|^2_{L^2(\boldsymbol{R}^m)}, \text{ if } s \text{ is an integer,} \\ \text{and} \\ \| v \|^2_{H^s(\boldsymbol{R}^m)} = \| u \|^2_{H^r(\boldsymbol{R}^m)} + \sum_{|\alpha|=r} \iint_{\boldsymbol{R}^m \times \boldsymbol{R}^m} \dfrac{| D^\alpha v(x) - D^\alpha v(y) |^2}{| x-y |^{m+2\theta}} dx\, dy \\ \qquad \text{if } s = r + \theta \text{ with } 0 < \theta < 1, \text{ and } r \text{ is an integer.} \end{cases}$$

Put

(5) $$I_\theta^2(u) = \iint_{\boldsymbol{R}_+^m \times \boldsymbol{R}_+^m} \frac{| u(x) - u(y) |^2}{| x-y |^{m+2\theta}} dx\, dy,$$

(6) $$J_\theta^2(u) = \iint_{\boldsymbol{R}_+^m \times \boldsymbol{R}_+^m} \frac{| u(x) + u(y) |^2}{| | x'-y' |^2 + | x_m + y_m |^2 |^{(m+2\theta)/2}} dx\, dy$$

and

(7) $$K_\theta^2(u) = \int_{\boldsymbol{R}_+^m} x_m^{-2\theta} | u(x) |^2 dx.$$

Then we have

(8) $$4\kappa_\theta^2 K_\theta^2(u) \le J_\theta^2(u) + I_\theta^2(u) \le I_\theta^2(u) + 4\kappa_\theta^2 K_\theta^2(u), \qquad 0 < \theta < 1,$$

where

$$\kappa_\theta^2 = \int_0^\infty (1+t)^{-1-2\theta} dt \int_{\boldsymbol{R}^{m-1}} \frac{dy'}{(| y' |^2 + 1)^{(m+2\theta)/2}}.$$

Replace v in (4) by λu, $u \in D(A_0^{1-\theta})$, and estimate the right hand side by means of (8) if necessary. Then, taking account of (3), we can establish Theorem 1 in the case of $\Omega = \boldsymbol{R}_+^m$ by a standard argument.

A similar argument can be applied in order to prove Theorem 2 in the case that $\Omega = \boldsymbol{R}_+^m$, and that α is identically equal to 1. Actually, instead of λ and μ, we use

$$\nu \colon L^2(\boldsymbol{R}_+^m) \to L^2(\boldsymbol{R}^m), \qquad \nu u(x', x_m) = \begin{cases} u(x', x_m), & x_m \ge 0 \\ u(x', -x_m), & x_m < 0, \end{cases}$$

$$\pi \colon L^2(\boldsymbol{R}^m) \to L^2(\boldsymbol{R}_+^m), \qquad \pi v(x', x_m) = \frac{1}{2} [\nu(x', x_m) + v(x', -x_m)] |_{\boldsymbol{R}_+^m}.$$

Then we resort to the following technic. Let $f(t)$ be a function in $C^\infty(\boldsymbol{R}^1)$ such that $\frac{1}{3} \le f(t) \le 3$, and $f(t) = t$ in some neighborhood of $t=1$. Since α never vanishes on $\boldsymbol{R}^{m-1}$ by the assumption, the map $\gamma \colon u \to f\left(\exp\left(-\frac{1-\alpha(x')}{\alpha(x')} x_m\right)\right) u$ gives isomorphism mappings

7) See footnote 6) and recall that
$$H^{s_1(1-\theta)+s_2\theta}(\Omega) = T(H^{s_1}(\Omega), H^{s_2}(\Omega); 2, \theta), \qquad 0 < \theta < 1.$$

$L^2(\boldsymbol{R}_+^m) \rightarrow L^2(\boldsymbol{R}_+^m)$ and $D(A_\alpha) \rightarrow D(A_1)$. Interpolating these,[8] we can reduce the case of general α to the special case of $\alpha \equiv 1$, which was already dealt with.

In the case of general bounded domain Ω with the smooth boundary, choose a tubular neighborhood U_1 of $\partial\Omega$ which is diffeomorphic to the product space $(-\delta, \delta) \times \partial\Omega$, for some $\delta > 0$. Using this diffeomorphism we can define maps λ', μ', ν', π', and γ', enjoying the properties similar to those of the maps λ, μ, ν, π, and γ. Therefore we can prove Theorems 1 and 2 in full generality by a slight modification of the argument used above in the case of $\Omega = \boldsymbol{R}_+^m$.

References

[1] T. Kato: Fractional powers of dissipative operators. J. Math. Soc. Japan, **13**, 246-274 (1961).

[2] J. L. Lions: Espaces d'interpolation et domaines de puissances fractionaires d'opérateurs. J. Math. Soc. Japan, **14**, 233-241 (1962).

[3] J. L. Lions et E. Magenes: Problème aux limites non homogènes. IV. Ann. Sc. Norm. Sup. Pisa, **15**, 311-326 (1961).

[4] E. Magenes: Spazi di interpolazione ed equazioni a derivate parziali. Atti del VII congresso dell'Unione Matematica Italiana, Geneva (1963).