18. *On the Absolute Logarithmic Summability of the Allied Series of a Fourier Series*

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1. **Introduction.** 1.1. **Definition.***) *Let $\lambda=\lambda(w)$ be continuous, differentiable and monotone increasing in $(0,\infty)$, and let it tend to infinity as $w\to\infty$. For a given series $\sum_1^\infty a_n$, we put*

$$C_r(w)=\sum_{n\le w}\{\lambda(w)-\lambda(n)\}^r a_n \qquad (r\geqslant 0).$$

Then the series $\sum_1^\infty a_n$ is called to be summable $|R,\lambda,r|$ $(r\geqslant 0)$, if

$$\int_A^\infty\left|d\left[\frac{C_r(w)}{(w)^r}\right]\right|<\infty \tag{1.1.1}$$

for a positive number A.

For $r>0$, and non-integral w, we have

$$\frac{d}{dw}\left[\frac{C_r(w)}{\{\lambda(w)\}^r}\right]=\frac{r\lambda'(w)}{\{\lambda(w)\}^{1+r}}\sum_{n\le w}\{\lambda(w)-\lambda(n)\}^{r-1}\lambda(n)a_n.$$

Hence $\sum_1^\infty a_n$ is summable $|R,\lambda,r|$ $(r>0)$, if and only if

$$\int_A^\infty\left|\frac{r\lambda'(w)}{\{\lambda(w)\}^{1+r}}\sum_{n\le w}\{\lambda(w)-\lambda(n)\}^{r-1}\lambda(n)a_n\right|dw<\infty. \tag{1.1.2}$$

1.2. We suppose that $f(t)$ is integrable in the Lebesgue sense in the interval $(-\pi,\pi)$, and is periodic with period 2π, so that

$$f(t)\sim\tfrac{1}{2}a_0+\sum_1^\infty(a_n\cos nt+b_n\sin nt)=\tfrac{1}{2}a_0+\sum_1^\infty A_n(t). \tag{1.2.1}$$

Then the allied series is

$$\sum_1^\infty(b_n\cos nt-a_n\sin nt)=\sum_1^\infty B_n(t). \tag{1.2.2}$$

We write

$$\psi(t)=\tfrac{1}{2}\{f(x+t)-f(x-t)\},\quad \theta(t)=\int_t^\pi\frac{\psi(u)}{u}du. \tag{1.2.3}$$

The object of the present paper is to prove the following

Theorem. *If $t^{-1}|\theta(t)|\log\frac{2\pi}{t}\in L(0,\pi)$, then* (1.2.2) *is summable $|R,\log w,2|$ at $t=x$.*

This theorem was conjectured by N. Basu in a stronger form.

2. **Proof of the Theorem.** 2.1. We write

$$g(w,t)=\sum_{n\le w}\log n\left(\log\frac{w}{n}\right)\sin nt, \tag{2.1.1}$$

*) Mohanty (1).

$$(2.1.2)\qquad h(w,t)=\sum_{n\leqslant w} n\log n\left(\log\frac{w}{n}\right)\cos nt.$$

For the proof of the theorem we require the following lemmas:

Lemma 1. $g(w,t)=O(w\log w)$.

Proof. By (2.1.1),

$$|g(w,t)|\leqslant\sum_{n\leqslant w}\log n\left(\log\frac{w}{n}\right).$$

Now we put

$$\sum_{n\leqslant w}\log n\left(\log\frac{w}{n}\right)=\sum_{n<w^{\frac{1}{2}}}\log n\left(\log\frac{w}{n}\right)+\sum_{w^{\frac{1}{2}}\leqslant n\leqslant w}\log n\left(\log\frac{w}{n}\right)=P+Q.$$

Since $\log u\left(\log\frac{w}{u}\right)$ is monotone increasing in $1\leqslant u<w^{\frac{1}{2}}$, we have, by the second mean value theorem,

$$P\leqslant\int_1^{w^{\frac{1}{2}}}\log u\left(\log\frac{w}{u}\right)du+O((\log w)^2)$$
$$\leqslant\log w^{\frac{1}{2}}(\log w^{\frac{1}{2}})w^{\frac{1}{2}}+O((\log w)^2)=O(w^{\frac{1}{2}}(\log w)^2).$$

Since $\log u\left(\log\frac{w}{u}\right)$ is monotone decreasing in $w^{\frac{1}{2}}\leqslant u\leqslant w$, we have

$$Q\leqslant\int_{w^{\frac{1}{2}}}^{w}\log u\left(\log\frac{w}{u}\right)du+O((\log w)^2)\leqslant\log w\int_{w^{\frac{1}{2}}}^{w}\left(\log\frac{w}{u}\right)du+O((\log w)^2)$$
$$=w\log w\int_1^{w^{\frac{1}{2}}}\frac{\log v}{v^2}\,dv+O((\log w)^2)=O(w\log w).$$

Hence we get the required inequality.

Lemma 2. $g(w,t)=O(t^{-1}(\log w)^2)$.

Proof. By Abel's lemma, we have

$$g(w,t)=\sum_{n\leqslant w}\log n\left(\log\frac{w}{n}\right)\sin nt$$
$$\leqslant\frac{A}{t}\sum_{n\leqslant w-1}\left|\Delta\left(\log n\log\frac{w}{n}\right)\right|+O(t^{-1}w^{-1}\log w)$$
$$\leqslant\frac{A}{t}\left\{\sum_{n<w^{\frac{1}{2}}}\left|\Delta\left(\log n\log\frac{w}{n}\right)\right|+\sum_{w^{\frac{1}{2}}\leqslant n\leqslant w-1}\left|\Delta\left(\log n\log\frac{w}{n}\right)\right|\right\}$$
$$+O(t^{-1}w^{-1}\log w)$$
$$=O(t^{-1}(\log w)^2),$$

since $\log u\left(\log\frac{w}{u}\right)$ is monotone increasing in $1\leqslant u<w^{\frac{1}{2}}$ and monotone decreasing in $w^{\frac{1}{2}}\leqslant u\leqslant w$.

Lemma 3. $g(w,t)=O(t^{-2}\log w)$.

Proof. Using Abel's lemma twice, we have

$$g(w,t)=\sum_{n\leqslant w}\log n\left(\log\frac{w}{n}\right)\sin nt$$
$$=\sum_{n\leqslant w-2}\Delta^2\left(\log n\log\frac{w}{n}\right)\sum_1^n\tilde{D}_v(t)+O(t^{-2}w^{-1}\log w)+O(t^{-1}w^{-1}\log w),$$

so that

$$(2.1.3)\qquad \left|g(w,t)\right| \leqslant \frac{A}{t^2}\sum_{n\leqslant w-2}\left|\Delta^2\left(\log n\log\frac{w}{n}\right)\right| + O(t^{-2}w^{-1}\log w)$$

$$=\frac{A}{t^2}\left\{\sum_{n<ew^{\frac{1}{2}}} + \sum_{ew^{\frac{1}{2}}\leqslant n\leqslant w-2}\right\} + O(t^{-2}w^{-1}\log w) = \frac{A}{t^2}(P+Q)+O(t^{-2}w^{-1}\log w).$$

Since $\left(\log u \log \dfrac{w}{u}\right)'$ is monotone decreasing in $1\leqslant u<ew^{\frac{1}{2}}$ and monotone increasing in $ew^{\frac{1}{2}}\leqslant u\leqslant w-2$, we have

$$(2.1.4)\qquad P=\sum_{n<ew^{\frac{1}{2}}}\left|\Delta\left(\frac{1}{n}\log\frac{w}{n}-\frac{1}{n}\log n\right)\right| = O(\log w),$$

$$(2.1.5)\qquad Q=\sum_{ew^{\frac{1}{2}}\leqslant n\leqslant w-2}\left|\Delta\left(\frac{1}{n}\log\frac{w}{n}-\frac{1}{n}\log n\right)\right| = O(w^{\frac{1}{2}}\log w).$$

Thus from (2.1.3), (2.1.4), and (2.1.5), we get the required inequality.

Lemma 4. $h(w,t)=O(t^{-1}w\log w)$.

Proof. Using Abel's lemma to (2.1.2), we get

$$h(w,t)=\sum_{n\leqslant w} n\log n\log\frac{w}{n}\cos nt$$

$$\leqslant \frac{A}{t}\sum_{n\leqslant w-1}\left|\Delta\left(n\log n\log\frac{w}{n}\right)\right| + O(t^{-1}\log w)$$

$$=\frac{A}{t}\left\{\sum_{n<\eta w}+\sum_{\eta w\leqslant n\leqslant w-1}\right\}+O(t^{-1}\log w)=\frac{A}{t}(P+Q)+O(t^{-1}\log w),$$

where η is taken so that $\left(u\log u\log\dfrac{w}{u}\right)$ is monotone increasing in $1\leqslant u<\eta w$ and monotone decreasing in $\eta w\leqslant u\leqslant w$, $0<\eta<1$. Therefore we get $P=O(w\log w)$ and $Q=O(w\log w)$. Hence we get the required inequality.

Lemma 5. $h(w,t)=O(t^{-2}(\log w)^2)$.

Proof. By twice use of Abel's lemma, we have

$$h(w,t)=\sum_{n\leqslant w-2}\Delta^2\left(n\log n\log\frac{w}{n}\right)\sum_{1}^{n}D_v(t)+O(t^{-2}\log w).$$

Therefore

$$|h(w,t)|\leqslant\frac{A}{t^2}\sum_{n\leqslant w-2}\left|\Delta^2\left(n\log n\log\frac{w}{n}\right)\right|+O(t^{-2}\log w)$$

$$=\frac{A}{t^2}\left\{\sum_{n<e^{-1}w^{\frac{1}{2}}}+\sum_{e^{-1}w^{\frac{1}{2}}\leqslant n\leqslant w-2}\right\}+O(t^{-2}\log w)=\frac{A}{t^2}(P+Q)+O(t^{-2}\log w).$$

Since $\left(u\log u\log\dfrac{w}{u}\right)'$ is monotone increasing in $1\leqslant u<e^{-1}w^{\frac{1}{2}}$ and is monotone decreasing in $e^{-1}w^{\frac{1}{2}}\leqslant u\leqslant w$, we have

$$P=\sum_{n<e^{-1}w^{\frac{1}{2}}}\left|\Delta\left(\log n\log\frac{w}{n}+\log\frac{w}{n}-\log n\right)\right|=O((\log w)^2),$$

$$Q=\sum_{e^{-1}w^{\frac{1}{2}}\leqslant n\leqslant w-2}\left|\Delta\left(\log n\log\frac{w}{n}+\log\frac{w}{n}-\log n\right)\right|=O((\log w)^2).$$

Hence we get the required inequality.

Lemma 6. $h(w,t)=O(t^{-3}\log w)$.

Proof. By three time use of Abel's lemma, we have

$$h(w,t)=\sum_{n\leqslant w-2}\Delta^2\left(n\log n\log\frac{w}{n}\right)\sum_1^n D_\nu(t)+O(t^{-2}\log w)$$

$$=\frac{1}{4\sin^2 t/2}\sum_{n\leqslant w-2}\Delta^2\left(n\log n\log\frac{w}{n}\right)$$

$$-\frac{1}{4\sin^2 t/2}\sum_{n\leqslant w-2}\Delta^2\left(n\log n\log\frac{w}{n}\right)\cos(n+1)t+O(t^{-2}\log w)$$

$$=-\frac{1}{4\sin^2 t/2}\sum_{n\leqslant w-2}\Delta^2\left(n\log n\log\frac{w}{n}\right)\cos(n+1)t+O(t^{-2}\log w)$$

$$=-\frac{1}{4\sin^2 t/2}\sum_{n\leqslant w-3}\Delta^3\left(n\log n\log\frac{w}{n}\right)\sum_1^n\cos(n+1)t+O(t^{-2}\log w).$$

Thus we have

$$|h(w,t)|\leqslant\frac{A}{t^3}\sum_{n\leqslant w-3}\left|\Delta^3\left(n\log n\log\frac{w}{n}\right)\right|+O(t^{-2}\log w)$$

$$=\frac{A}{t^3}\left\{\sum_{n<w^{\frac12}}+\sum_{w^{\frac12}\leqslant n\leqslant w-3}\right\}+O(t^{-2}\log w)=\frac{A}{t^3}(P+Q)+O(t^{-2}\log w).$$

Since $\left(u\log u\log\frac{w}{u}\right)''$ is monotone decreasing in $1\leqslant u<w^{\frac12}$ and monotone increasing in $w^{\frac12}\leqslant u\leqslant w$, we have

$$P=\sum_{n<w^{\frac12}}\left|\Delta\left(\frac{1}{n}\log\frac{w}{n}-\frac{1}{n}\log n-\frac{2}{n}\right)\right|=O(\log w).$$

Similarly we have $Q=O(w^{-\frac12}\log w)$. Hence we get the required inequality.

2.2. We shall now prove the theorem. By integrating by parts, we find

$$(2.2.1)\qquad B_n(x)=\frac{2}{\pi}\int_0^\pi\psi(t)\sin nt\,dt=-\frac{2}{\pi}\int_0^\pi t\theta'(t)\sin nt\,dt$$

$$=\frac{2}{\pi}\int_0^\pi\theta(t)\sin nt\,dt+\frac{2}{\pi}\int_0^\pi\theta(t)\,nt\cos nt\,dt.$$

The series $\sum_1^\infty B_n(x)$ is summable $|R,\log w,2|$, if

$$I=\int_e^\infty\frac{dw}{w(\log w)^3}\left|\sum_{n\leqslant w}\log n\log\frac{w}{n}B_n(x)\right|<\infty.$$

Substituting (2.2.1) for $B_n(x)$, we have, by (2.1.1) and (2.1.2),

$$(2.2.2)\qquad I\leqslant\frac{2}{\pi}\int_0^\pi|\theta(t)|\,dt\int_e^\infty w^{-1}(\log w)^{-3}|g(w,t)|\,dw$$

$$+\frac{2}{\pi}\int_0^\pi t|\theta(t)|\,dt\int_e^\infty w^{-1}(\log w)^{-3}|h(w,t)|\,dw.$$

Since $\int_0^\pi t^{-1}|\theta(t)|\log\frac{2\pi}{t}\,dt$ is finite, it is enough to show that

$$I_1=\int_e^\infty w^{-1}(\log w)^{-3}\,|\,g(w,t)\,|\,dw=O\Big(t^{-1}\log\frac{2\pi}{t}\Big)\qquad \text{for } 0<t<\pi;$$

and

$$I_2=\int_e^\infty w^{-1}(\log w)^{-3}\,|\,h(w,t)\,|\,dw=O\Big(t^{-2}\log\frac{2\pi}{t}\Big)\qquad \text{for } 0<t<\pi.$$

Let $A_1=\frac{2\pi}{t}\log\frac{2\pi}{t}$, $A_2=e^{2\pi/t}$ and let

$$I_1=\int_e^{A_1}+\int_{A_1}^{A_2}+\int_{A_2}^\infty=I_{11}+I_{12}+I_{13}. \tag{2.2.3}$$

Using Lemma 1, we have

$$I_{11}=O\Big(\int_e^{A_1}(\log w)^{-2}dw\Big)=O\Big(t^{-1}\Big(\log\frac{2\pi}{t}\Big)^{-1}\Big). \tag{2.2.4}$$

By Lemma 2, we have

$$I_{12}=O\Big(t^{-1}\int_{A_1}^{A_2}w^{-1}(\log w)^{-1}dw\Big)=O\Big(t^{-1}\log\frac{2\pi}{t}\Big). \tag{2.2.5}$$

By Lemma 3, we have

$$I_{13}=O\Big(t^{-2}\int_{A_2}^\infty w^{-1}(\log w)^{-2}dw\Big)=O(t^{-1}). \tag{2.2.6}$$

Hence from (2.2.3), (2.2.4), (2.2.5), and (2.2.6), we get $I_1=O\Big(t^{-1}\log\frac{2\pi}{t}\Big)$.

It remains to prove that $I_2=O\Big(t^{-2}\log\frac{2\pi}{t}\Big)$. Let $A_1=\frac{2\pi}{t}\log\frac{2\pi}{t}$, $A_2=e^{2\pi/t}$ and let

$$I_2=\int_e^\infty=\int_e^{A_1}+\int_{A_1}^{A_2}+\int_{A_2}^\infty=I_{21}+I_{22}+I_{23}. \tag{2.2.7}$$

By Lemma 4, we have

$$I_{21}=O\Big(t^{-1}\int_e^{A_1}(\log w)^{-2}dw\Big)=O\Big(t^{-2}\Big(\log\frac{2\pi}{t}\Big)^{-1}\Big). \tag{2.2.8}$$

By Lemma 5, we have

$$I_{22}=O\Big(t^{-2}\int_{A_1}^{A_2}w^{-1}(\log w)^{-1}dw\Big)=O\Big(t^{-2}\log\frac{2\pi}{t}\Big). \tag{2.2.9}$$

By Lemma 6, we have

$$I_{23}=O\Big(t^{-3}\int_{A_2}^\infty w^{-1}(\log w)^{-2}dw\Big)=O(t^{-2}). \tag{2.2.10}$$

Hence from (2.2.7), (2.2.8), (2.2.9), and (2.2.10), we have the required inequalily for I_2. Thus the proof of the theorem is completed.

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Reference

[1] R. Mohanty: On the absolute Riesz summability of a Fourier series and its allied series. Proc. London Math. Soc., **52**(2), 295-320 (1951).