68. *On Extensions of Automorphisms of Abelian von Neumann Algebras*

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1. Let $\mathcal{A}$ be a maximal abelian von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, ϕ a faithful normal trace with a normalized trace vector and G a countable freely acting ergodic group of ϕ-preserving automorphisms of $\mathcal{A}$. Then we can raise the following questions with respect to automorphisms of $\mathcal{A}$ and automorphisms of the crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G.

1) What kind of automorphisms of $\mathcal{A}$ can be extended to what kind of automorphisms of $G\otimes\mathcal{A}$?

2) Especially, what kind of automorphisms of $\mathcal{A}$ can be extended to inner automorphisms of $G\otimes\mathcal{A}$?

3) What kind of unitary operators in $G\otimes\mathcal{A}$ induce inner automorphisms of $G\otimes\mathcal{A}$ which preserve $\mathcal{A}$?

4) How does the questions 1) or 2) depend on the properties of G?

In this paper, the questions 1) and 4) will be discussed according to several conditions. The questions 2) and 3) are already discussed in [1] and [4].

Hereafter, we assume all automorphisms of $\mathcal{A}$ are ϕ-preserving *-automorphisms, and the terminology and the notations of [2] will be employed without further explanations.

2. We shall reformulate a theorem of I. M. Singer [5; Lemma 2.2] using the terminology of the crossed product:

Theorem 1. *Let $\mathcal{A}$ be a maximal abelian von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, ϕ a faithful normal trace with a normalized trace vector, G a countable freely acting ergodic group of automorphisms of $\mathcal{A}$ and σ an inner automorphism of $G\otimes\mathcal{A}$ such that $\mathcal{A}^\sigma=\mathcal{A}$.*

Then σ is induced by a unitary operator

$$U=\sum_{g\in G} VE_gU_g,$$

where V and E_g satisfy the following conditions:

(1) *V is a unitary operator in $\mathcal{A}$,*

(2) *E_g is a projection in $\mathcal{A}$ for each $g\in G$,*

(3) *$E_gE_h=0$ for $g\neq h$,*

(4) *$\sum_{g\in G} E_g=1$,*

(5) *E_g is absolutely fixed under αg^{-1}, where α is a restriction of σ in $\mathcal{A}$.*

The projection E_g in Theorem 1 is equal to the projection $F(\alpha, g)$ in $\mathcal{A}$ which is a maximal projection absolutely fixed under αg^{-1}, cf. [2].

A counterpart of Theorem 1 for finite factors is discussed in [4].

In what follows, the notations in Theorem 1 will be employed throughout.

3. In this section, we shall discuss the question 1). As well known, any (ϕ-preserving) automorphism α of $\mathcal{A}$ can be extended to an automorphism θ of $G\otimes\mathcal{A}$. However, it is not obvious that there exists a desired extension of α if θ is restricted by certain conditions. We shall give an answer for a very restrictive one:

Theorem 2. *Let $\mathcal{A}$ and G be same as in Theorem 1, and α an automorphism of $\mathcal{A}$. Then α can be extended to an automorphism θ of $G\otimes\mathcal{A}$ such that, for each $g\in G$,*

$$U_g^\theta = U_h \qquad \text{for some } h\in G,$$

if and only if α satisfies

$$\alpha^{-1}G\alpha = G.$$

Proof. If α can be extended to an automorphism θ of $G\otimes\mathcal{A}$ which satisfies the requirement of the theorem, then h depends on g, that is, $h=\varphi(g)$, and φ is an automorphism of G because θ is an automorphism. Since

$$\begin{aligned} U_{\varphi(h)}A^{h^{-1}\alpha} &= (U_h A^{h^{-1}})^\theta = (AU_h)^\theta \\ &= A^\alpha U_{\varphi(h)} = U_{\varphi(h)} A^{\alpha\varphi(h^{-1})}, \end{aligned}$$

for any $A\in\mathcal{A}$ and $h\in G$, we have $\varphi(h^{-1})=\alpha^{-1}h^{-1}\alpha$. Hence

$$\varphi(h)=\alpha^{-1}h\alpha,$$

for any $h\in G$. Therefore, $\alpha^{-1}G\alpha\subset G$. Since θ^{-1} is an extension of α^{-1}, a similar computation shows that $\alpha G\alpha^{-1}\subset G$. Hence we have $\alpha^{-1}G\alpha=G$.

Conversely, let $\alpha^{-1}G\alpha=G$, then we can define an automorphism α of G by

$$\varphi(g)=\alpha^{-1}g\alpha.$$

Using this automorphism φ, define the mapping θ by

$$(AU_g)^\theta = A^\alpha U_{\varphi(g)} \qquad \text{for any } g\in G \text{ and } A\in\mathcal{A}.$$

On the other hand, if we define

$$U'(g\otimes A)=\varphi(g)\otimes A^\alpha,$$

and

$$U'\Big(\sum_{i=1}^n g_i\otimes A_i\Big)=\sum_{i=1}^n U'(g_i\otimes A_i),$$

then the mapping U' can be extended to a unitary operator U on $G\otimes\mathfrak{H}$. And we have

$$\begin{aligned}UAU_hU^*(g\otimes B) &= UAU_h(\varphi^{-1}(g)\otimes B^{\alpha^{-1}}) \\ &= U(h\varphi^{-1}(g)\otimes AB^{\alpha^{-1}h^{-1}}) \\ &= A^\alpha U_{\varphi(h)}(g\otimes B) \\ &= (AU_h)^\theta(g\otimes B),\end{aligned}$$

for any $g\in G$ and $A, B\in\mathcal{A}$. Therefore,

$$(AU_g)^\theta = UAU_gU^*.$$

Hence α can be extended to an automorphism of $G\otimes\mathcal{A}$.

4. In this section, we shall discuss the question 4).

Theorem 3. *Let $\mathcal{A}$, G, σ, V, and E_g be as in Theorem* 1. *Let φ be an automorphism of G. Then the following conditions are equivalent*:

(6) $\quad U_g^\sigma = U_{\varphi(g)}$ *for every* $g\in G$,

(7) $\quad (VE_h)^g = VE_{g^{-1}h\varphi(g)}$ *for every g and h in G.*

Proof. Assume (6). Then

$$U_gU = UU_{\varphi(g)}.$$

By direct computations, we have

$$U_gU = U_g\Big(\sum_{h\in G} VE_hU_h\Big) = \sum_{h\in G}(VE_{g^{-1}h\varphi(g)})^{g^{-1}}U_{h\varphi(g)}$$

and $$UU_{\varphi(g)} = \Big(\sum_{h\in G} VE_hU_h\Big)U_{\varphi(g)} = \sum_{h\in G} VE_hU_{h\varphi(g)}.$$

Hence we have

$$\sum_{h\in G}(VE_{g^{-1}h\varphi(g)})^{g^{-1}}U_{h\varphi(g)} = \sum_{h\in G} VE_hU_{h\varphi(g)}.$$

Comparing the coefficients of $U_{h\varphi(g)}$ in the both sides, we have (7).

Conversely, suppose (7). Then we have

$$UU_{\varphi(g)} = \sum_{h\in G} VE_hU_{h\varphi(g)} = \sum_{h\in G}(VE_{g^{-1}h\varphi(g)})^{g^{-1}}U_{h\varphi(g)} = U_gU.$$

This proves the theorem.

Theorem 4. *If the set $I_g=\{hgh^{-1}; h\in G\}$ is infinite for each $g\in G, g\neq 1$, then the automorphism $\sigma, \sigma\neq 1$, of $G\otimes\mathcal{A}$ such that $U_g^\sigma = U_g$ for every $g\in G$ is outer.*

Proof. Suppose that σ is an inner automorphism of $G\otimes\mathcal{A}$. Then, since σ preserves the algebra $\mathcal{A}$ by $U_g^\sigma = U_g$, σ is induced by

$$U = \sum_{g\in G} VE_gU_g$$

of Theorem 1. Therefore, by Theorem 3, we have

(8) $\quad (VE_h)^g = VE_{g^{-1}hg}$ *for every g and h in G.*

Putting $h=1$ in (8), we have

$$(VE_1)^g = VE_1,$$

for every $g\in G$. Therefore, by the ergodicity of G, VE_1 is a scalar multiple of the identity. Hence E_1 is either 1 or 0.

Now, we shall divide the proof in two cases:

Case 1. Suppose $E_1=1$. Then $E_g=0$ for all $g\neq 1$ by (4). Hence $U=VU_1=V$, so that $\sigma=1$, which contradicts the hypothesis that $\sigma\neq 1$.

Case 2. Suppose $E_1=0$. Then $E_h\neq 0$ for some $h\neq 1$. By (8),

$$E_{g^{-1}hg}=V^*(VE_h)^g$$

for each $g\in G$, and

$$\|E_{g^{-1}hg}\|_2^2=\|V^*(VE_h)^g\|_2^2=\|E_h\|_2^2,$$

where $\|A\|_2^2=\phi(A*A)$ for any $A\in\mathcal{A}$. Since I_h is infinite, we have

$$1=\|\sum_{g\in G}E_g\|_2^2\geqq\|\sum_{k\in I_h}E_k\|_2^2=\sum_{k\in I_h}\|E_k\|_2^2=+\infty,$$

which is a contradiction.

Theorem 5. *If G is an abelian group and if an inner automorphism σ of $G\otimes\mathcal{A}$ satisfies $U_g^\sigma=U_g$ for every $g\in G$, then σ is induced by U_h for some $h\in G$.*

Proof. Suppose that σ is induced by

$$U=\sum_{g\in G}VE_gU_g$$

of Theorem 1. By Theorem 3, $U_g^\sigma=U_g$ implies

$$(VE_h)^g=VE_{g^{-1}hg}=VE_h$$

for every g and h in G. By the ergodicity of G, we have

$$VE_h=a_h1$$

for each $h\in G$, where a_h is a scalar. Hence $E_h=0$ or 1 for each $h\in G$. Therefore, we have

$$U=\sum_{g\in G}VE_gU_g=a_hU_h,$$

for some $h\in G$. Consequently, σ is induced by U_h.

5. Before to conclude the note, we shall discuss a relation between a certain group G and its full group $[G]$ introduced by H. A. Dye [3].

Theorem 6. *If G is an abelian group which is ergodic and freely acting on $\mathcal{A}$, then G is a maximally abelian subgroup in the full group $[G]$ determined by G.*

Proof. Let α be an automorphism in $[G]$ such that $\alpha g=g\alpha$ for every $g\in G$. Then by [1; Theorem 1] α can be extended to an inner automorphism of $G\otimes\mathcal{A}$ which is induced by a unitary operator

$$U=\sum_{g\in G}E_gU_g,$$

where $E_g=F(\alpha,g)$.

For any projection $Q\leqq E_g^h$, we have $Q^{h^{-1}}\leqq E_g$, so that

$$Q^{h^{-1}}=Q^{h^{-1}\alpha g^{-1}}=Q^{\alpha g^{-1}h^{-1}}.$$

Hence $Q^\alpha=Q^g$. Therefore E_g^h is absolutely fixed under αg^{-1}, and so $E_g^h\leqq E_g$. Since h is ϕ-preserving and since ϕ is faithful, we have $E_g^h=E_g$ for every g and h in G. Therefore $E_g=0$ or 1 for every $g\in G$ since G is ergodic. By (4) of Theorem 1, we have $U=U_g$ for some $g\in G$. This completes the proof of the theorem.

References

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