58. On the Geometry of G-Structures of Higher Order

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Let $V=R^n$ and V^* its dual. Let M be a differentiable manifold of dimension n and $F^r(M)$ the bundle of r-frames of M. The structure group of $F^r(M)$ is denoted by $G^r(n)$. The Lie algebra $\mathfrak{g}^r(n)$ of $G^r(n)$ is $V\otimes V^*+V\otimes S^2(V^*)+\cdots+V\otimes S^r(V^*)$.

A *transitive graded Lie algebra* is, by definition, a Lie subalgebra $\tilde{\mathfrak{g}}=V+\mathfrak{g}_0+\mathfrak{g}_1+\cdots$ of $V+V\otimes V^*+V\otimes S^2(V^*)+\cdots$, with $\mathfrak{g}_i\subset V\otimes S^{i+1}(V^*)$, satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j]\subset\mathfrak{g}_{i+j}$$

where $\mathfrak{g}_{-1}=V$.

We call that $\tilde{\mathfrak{g}}$ is of *order* r if

$$\mathfrak{g}_{i+j}\subsetneqq\mathfrak{g}_i^{(j)} \qquad \text{for } i+j<r$$

and

$$\mathfrak{g}_{i+j}=\mathfrak{g}_i^{(j)} \qquad \text{for } i\geqq r \text{ and } j\geqq 0.$$

If $\mathfrak{g}_{k-1}\neq 0$ and $\mathfrak{g}_k=0$ then $\tilde{\mathfrak{g}}$ is said to be of *type* k. In general $r\leqq k+1$.

Let $M_0=\tilde{G}/G$ be a homogeneous space of dimension n. Suppose $\tilde{G}$ is a finite dimensional Lie group whose Lie algebra $\tilde{\mathfrak{g}}$ is a transitive graded Lie algebra of order r and of type k:

$$\tilde{\mathfrak{g}}=V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{s-1}$$

where $s=\text{Max}\{r, k\}$.

We also suppose that G is a closed subgroup of $\tilde{G}$ whose Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1+\cdots+\mathfrak{g}_{s-1}.$$

Then G can be considered as a subgroup of $G^s(n)$.

Definition. Let M be a differentiable manifold of dimension n and G a subgroup of $G^s(n)$ as above. A G-structure $P_G(M)$ of order r and of type k on M is a reduction of $F^s(M)$ to the group G.

Example 1. *Affine structure.* Let $\tilde{G}$ be the affine group and G the isotropy subgroup at the origin so that $\tilde{G}/G$ is the affine space. Then $\tilde{\mathfrak{g}}=V+\mathfrak{gl}(n)=V+V\otimes V^*$ and $\mathfrak{g}=\mathfrak{gl}(n)$. An affine structure on M is, by definition, a reduction of $F^2(M)$ to the group G. Affine structure is a G-structure of order 2 and of trpe 1.

Example 2. *Projective s*'*tructu*, *e.* Let $\tilde{G}$ be the group of projective transformations of a real projective space of dimension n and G the isotropy subgroup at the distinguished point so that $\tilde{G}/G$ is the real projective space. Let $\mathfrak{p}\cong V^*$ be the invariant complement to $\mathfrak{sl}(n)^{1)}$ in $\mathfrak{gl}(n)^{1)}$. Then

$$\tilde{\mathfrak{g}}=V+\mathfrak{gl}(n)+\mathfrak{p} \text{ and } \mathfrak{g}=\mathfrak{gl}(n)+\mathfrak{p}.$$

A projective structure on M is, by definition, a reduction of $F^2(M)$ to the group G. Projective structure is a G-structure of order 2 and of type 2.

Example 3. *Conformal structure.*

Let $\tilde{G}$ be the group of Möbius transformations of a Möbius space of dimension n and G the isotropy subgroup at a point so that $\tilde{G}/G$ is the Möbius space. Then $\tilde{\mathfrak{g}}=V+\mathfrak{co}(n)+\mathfrak{co}(n)^{1)}\cong V+\mathfrak{co}(n)+V^*$ and $\mathfrak{g}=\mathfrak{co}(n)+\mathfrak{co}(n).^{1)}$ A conformal structure on M is, by definition, a reduction of $F^2(M)$ to the group G. Conformal structure is a G-structure of order 1 and of type 2.

Let $P_G(M)$ be a G-structure of order r and of type k on M. Let θ be the canonical form of $F^s(M)$ restricted to $P_G(M)$. Then θ is a $V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{s-2}$-valued 1-form on $P_G(M)$. Let ω_i be the $\mathfrak{g}_i$-component of θ, then $\theta=(\omega_{-1}, \omega_0, \omega_1, \cdots, \omega_{s-2})$. For each $u\in P_G(M)$, let G_u be the subspace of $T_u(P_G(M))$ consisting of vectors tangent to the fibre through u. Then $G_u\cong\mathfrak{g}$. *A* complement to G_u in $T_u(P_G(M))$ on which the forms $\omega_0, \omega_1, \cdots, \omega_{s-2}$ all vanish is called a *horizontal space* at u. Let H be a horizontal space at u, then $H\cong V$. Now let ξ and η be elements of V, and X and Y the corresponding elements in H. We define

$$c_H\in \mathrm{Hom}(V\wedge V,\ V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{s-2})$$

by

$$c_H(\xi, \eta)=d\theta(X, Y).$$

We shall denote the $\mathrm{Hom}(V\wedge V, \mathfrak{g}_i)$-component of c_H by c^i_H. Then c^i_H is a cocycle. Let H and H' be two horizontal spaces at$u$. It is easily seen that

$$c^i_{H'}-c^i_H\in\partial\,\mathrm{Hom}(V, \mathfrak{g}_{i+1}) \quad \text{for} \quad i=-1, 0, 1, \cdots, s\text{-}2.$$

Hence the cohomology class c^i of c^i_H is independent of the choice of the horizontal space H. c^i is an element of the Spencer cohomology group $H^{i+1,2}$ associated with the bigraded chain complex

$$\sum_{i,j}\mathfrak{g}_{i-1}\otimes\wedge^j(V^*).$$

We call $c=(c^{-1}, c^0, c^1, \cdots, c^{s-2})$ the *structure tensor* of the G-structure $P_G(M)$. c is a $\sum_{i=0}^{s-1}H^{i,2}$-valued function on $P_G(M)$. $P_G(M)$ is said to be *l-flat* if $c^i=0$ for $i\leq l\text{-}2$.

$\tilde{G}$ operates transitively on M_0 and G can be considered as the isotropy subgroup at a point of M_0 so that $M_0=\tilde{G}/G$. M_0 has a natural G-structure. The G-structure is called the *standard flat* G-structure.

A G-structure is said to be *flat* if it is locally isomorphic with the standard flat G-structure.

If $s=k+1$ we set $G'=G$. If $s=k$, let G' be a semidirect product of G and the nilpotent Lie group generated by $\mathfrak{g}_k+\mathfrak{g}_{k+1}+\cdots/\mathfrak{g}_{k+1}+\cdots$. Then G' can be considered as a subgroup of $G^{k+1}(n)$ and whose image under the projection $G^{k+1}(n)\to G^s(n)$ is just G.

There exists a reduction of $F^{k+1}(M)$ to G' which is identical with $P_G(M)$. We shall denote the reduced bundle by $P'_G(M)$. Let θ' be the canonical form of $F^{k+1}(M)$ restricted to $P'_G(M)$. Then θ' is a $V+\mathfrak{g}_0+\cdots+\mathfrak{g}_{k-1}$-valued 1-form on $P'_G(M)$. Let c' be the structure tensor of $P'_G(M)$. Then $c'=(c^{-1}, c^0, c^1, \cdots, c^{k-2}, c^{k-1})$, that is, $H^{i,2}$-components of c' for $i\leqq s$-2 are identical with those of c.

Theorem 1. *A G-structure $P_G(M)$ of order r and of type k is flat if and only if it is $(k+1)$-flat, that is, $c'=0$.*

Let $P_G(M)$ be a G-structure of order r and of type k and $\mathfrak{L}$ the sheaf of germs of infinitesimal automorphisms of $P_G(M)$. Let $\mathfrak{L}_x$ be the stalk at $x\in M$. Then $\dim \mathfrak{L}_x\leqq\dim P_G(M)$. We have the following

Theorem 2. *Let $P_G(M)$ be a G-structure of order r and of type k on M. Suppose $\mathfrak{A}_0$ contains the identity element. Then $P_G(M)$ is flat if and only if* $\dim L_x=\dim P_G(M)$ *at every point x of M.*

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