106. *On Normal Analytic Sets*

By Ikuo KIMURA

Kōbe University

(Comm. by Kinjirô KUNUGI, M.J.A., June 12, 1967)

In this paper, we shall show that an irreducible analytic set at a point is always described locally by a certain number of systems of Puiseux-series. And we shall present a theorem saying that an irreducible analytic set in a neighborhood of a point is *normal* if and only if such a group of systems of Puiseux-series satisfies the following conditions.

(1) Two systems of the group never pass through any common point.

(2) If the order of the series of a system belonging to the group exceeds 1, the second coefficient of a series of the system does not vanish identically.

We shall further give a theorem concerning the dimension of the set of non-normal points.

We suppose that the analytic sets are in the space of n complex variables and of d-dimension at the point we consider, where n surpasses 2 and d surpasses 1—the reason of which is that, if $d=1$, all the circumstances reduce to a very clear situation and our results subsist without any alteration.

1. Representation by systems of Puiseux-series. We work in the space of n variables $x_1, \cdots, x_n (n>2)$. Let $\sum$ be an analytic set in an open set containing a point A; we suppose that $\sum$ is *irreducible at* A. For brevity, we assume that A is the origin. Then by the local description theorem,[1] if we choose a proper system of coordinates, there exist a polydisc

$$C: |x_i|<r_i, 1\leqslant i\leqslant n,$$

with arbitrarily small radii r_i, a distinguished pseudo-polynomial of degree an integer N in x_{d+1}:

$$P(x_1, \cdots, x_{d+1})=x_{d+1}^N+a_1(x_1, \cdots, x_d)x_{d+1}^{N-1}+ \cdots +a_N(x_1, \cdots, x_d) ,$$

and, for each $i, d+1<i\leqslant n$, a pseudo-polynomial of degree $\leqslant N-1$ in x_{d+1}:

$$Q_i(x_1, \cdots, x_{d+1})=b_1^{(i)}(x_1, \cdots, x_d)x_{d+1}^{N-1}+\cdots+b_N^{(i)}(x_1, \cdots, x_d) ,$$

such that they together satisfy the following conditions.

(1) The coefficients $a_j(x_1, \cdots, x_d)$, $b_j^{(i)}(x_1,\cdots, x_d)$ of P, Q_i are holomorphic in the polydisc

1) M. Hervé: Several Complex Variables, Tata Institute of Fundamental Research, Bombay, Oxford University Press, 1963.

$$\Gamma: |x_i| < r_i,\ 1 \leqslant i \leqslant d,$$

and, but for the leading coefficient of P, vanish at the origin.

(2) $P(x_1, \cdots, x_{d+1})$ is irreducible at the origin.

(3) For each point $(x_1, \cdots, x_d) \in \Gamma$, all the N roots x_{d+1} of the equation $P=0$ lie in the disc $|x_{d+1}| < r_{d+1}$.

(4) Let $D(x_1, \cdots, x_d)$ be the discriminant of P. Then the set of the points belonging to $\sum$ and satisfying $D(x_1, \cdots, x_d) \neq 0$, is represented in C, by the conditions:

$$(x_1, \cdots, x_d) \in \Gamma,\ D(x_1, \cdots, x_d) \neq 0,\ P(x_1, \cdots, x_{d+1}) = 0,$$

$$x_i = \frac{Q_i(x_1, \cdots, x_{d+1})}{\partial P(x_1, \cdots, x_{d+1})/\partial x_{d+1}},\ d+1 < i \leqslant n.$$

Moreover, we have to be observant on the fact that this local expression is also possible, if we do a linear regular transformation of the coordinates $x_1, \cdots, x_d$, or even if we do a linear and sufficiently small transformation of the coordinates $x_{d+1}, \cdots, x_n$. Consequently we may assume, without loss of generality, that we have

$$D(x_1, \cdots, x_d) = E(x_1 \cdots, x_d) W(x_1, \cdots, x_d),$$

where E is a holomorphic and non-zero function in a neighborhood of the origin, and W a distinguished pseudo-polynomial in x_d:

$$W(x_1, \cdots, x_d) = x_d^M + c_1(x_1, \cdots, x_{d-1}) x_d^{M-1} + \cdots + c_M(x_1, \cdots, x_{d-1}).$$

Let

$$W(x_1, \cdots, x_d) = f_1(x_1, \cdots, x_d)^{q_1} \cdots f_s(x_1, \cdots, x_d)^{q_s};$$

be the decomposition of W into distinguished irreducible factors at the origin. Let

$$f(x_1, \cdots, x_d) = f_1(x_1, \cdots, x_d) \cdots f_s(x_1, \cdots, x_d);$$

then f is a distinguished pseudo-polynomial in x_d:

$$f(x_1, \cdots, x_d) = x_d^L + h_1(x_1, \cdots, x_{d-1}) x_d^{L-1} + \cdots + h_L(x_1, \cdots, x_{d-1}),$$

where h_i are holomorphic and zero at the origin. Taking C small enough, we may add to the above theorem following two conditions.

(5) $E(x_1, \cdots, x_d)$ is holomorphic and non-zero in Γ. The $h_i(x_1, \cdots, x_{d-1})$ are holomorphic in the polydisc

$$\gamma: |x_i| < r_i,\ 1 \leqslant i < d.$$

(6) The relations $(x_1, \cdots, x_{d-1}) \in \gamma,\ f(x_1, \cdots, x_d) = 0$ induce $(x_1, \cdots, x_d) \in \Gamma$.

These six assumptions for $\sum$ explained above are not disturbed, if we do a small linear change of the coordinates $x_{d+1}, \cdots, x_n$. In the following, we discuss mainly the case $L \geqslant 1$, because, if $f=1$, we see that our results subsist under the same form.

Finally we explain the notations employed in this paper.

(a) Let $k = d-1,\ e = n-d$.

(b) The variables $x_d, x_{d+1}, \cdots, x_n$ are denoted by $y, z_1, \cdots, z_e$, respectively.

(c) The points $(x_1, \cdots, x_k)$, $(x_1, \cdots, x_k, y)$, $(x_1, \cdots, x_k, y, z_1, \cdots, z_e)$ are denoted by x, (x, y), (x, y, z), respectively.

(d) The origin of a space of arbitrary number of variables is denoted simply by the zero 0.

Let give an analytical expression for $\sum$ in the neighborhood of the variety $f(x, y)=0$. Let x^0 be a point in γ, at which $\delta(x)$ does not vanish. Then the equation $f(x^0, y)=0$ has L simple zeros: let y^0 be one of them. At that time, the equation $f(x, y)=0$ has a unique solution $y=\varphi(x)$ holomorphic in a neighborhood of x^0 and such that $y^0=\varphi(x^0)$. Consequently, in a neighborhood of (x^0, y^0), the solutions of the equation $P(x, y, z_1)=0$ are represented by a certain number κ of Puiseux-series:

$$z_1=z_1^{(i)}(x, y)=\sum_{\nu=0}^{\infty} c_\nu^{(i)}(x)(y-\varphi(x))^{\frac{\nu}{p_i}},\ 1\leqslant i\leqslant \kappa,$$

where $c_\nu^{(i)}(x)$ are holomorphic in a neighborhood of x^0. Putting the expressions above into the formulas given in (4), we see that the z_i, $1<i\leqslant e$ are also represented by Puiseux-series.

Accordingly, we have a certain number κ of systems of Puiseux-series:

$$z_\mu=z_\mu^{(i)}(x, y)=\sum_{\nu=0}^{\infty} c_\nu^{(i,\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p_i}},\ 1\leqslant \mu\leqslant e,\ 1\leqslant i\leqslant \kappa,$$

where $c_\nu^{(i,\mu)}(x)$ are holomorphic in a neighborhood of x^0; this group of systems of Puiseux-series describes $\sum$ completely in a neighborhood of (x^0, y^0).

2. A condition for local irreducibility. As to the local irreducibility of $\sum$, we have in the first place the following

Proposition 1. *Let (x^0, y^0, z^0) be a point on $\sum\cap C$. Then $\sum$ is irreducible at (x^0, y^0, z^0), if and only if $P(x, y, z_1)$ is irreducible at (x^0, y^0, z_1^0).*

Let (x^0, y^0) be a point in Γ, satisfying $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$. Let

$$(1)\qquad z_\mu=z_\mu^{(i)}(x, y)=\sum_{\nu=0}^{\infty} c_\nu^{(i,\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p_i}},\ 1\leqslant \mu\leqslant e,\ 1\leqslant i\leqslant \kappa,$$

be the systems of Puiseux-series representing $\sum$ in a neighborhood of (x^0, y^0). We consider the condition below.

Condition (α). *If the point (x^0, y^0) is sufficiently near* 0, *then, for i, j, $i\neq j$, we have $c_0^{(i,1)}(x)\neq c_0^{(j,1)}(x)$ for the points x near x^0.*

Remarks. (a) If (x^0, y^0, z^0) is a point sufficiently near 0 such that $\delta(x^0)\neq 0$, $f(x^0, y^0)=0$, and if $\sum$ satisfies the condition (α), then there exists a unique system of (1) passing through the point (x^0, y^0, z^0) and therefore representing $\sum$ completely in a neighborhood of (x^0, y^0, z^0). (b) The functions $c_0^{(i,1)}(x)$ have the property that, if $c_0^{(i,1)}(x)\not\equiv c_0^{(j,1)}(x)$, we have $c_0^{(i,1)}(x)\neq c_0^{(j,1)}(x)$ for x near x^0, or equivalently that, if $c_0^{(i,1)}(x)=c_0^{(j,1)}(x)$ for a point x near x^0, we have necessarily

$c_0^{(i,1)}(x) \equiv c_0^{(j,1)}(x)$. This is a consequence of a lemma due to K. Oka.[2)]

We have the following

Proposition 2. *$\sum$ is locally irreducible at 0, if and only if $\sum$ satisfies the condition (α).*

3. Conditions for normality. As a necessary condition for normality, we have the

Proposition 3. *If $\sum$ is normal at 0, the set $\sum$ satisfies the condition (α).*

Now we consider an another condition:

Condition (β). Let (x^0, y^0) be a point sufficiently near 0, such that $\delta(x^0) \neq 0, f(x^0, y^0) = 0$. And let

$$z_\mu = z_\mu(x, y) = \sum_{\nu=0}^{\infty} c_\nu^{(\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p}},\ 1 \leqslant \mu \leqslant e,$$

be a system of Puiseux-series, attached to the point (x^0, y^0), such that $p>1$; then we have $c_1^{(\mu)}(x) \not\equiv 0$ for an index μ, $1 \leqslant \mu \leqslant e$.

The above condition is also necessary for normality:

Proposition 4. *If $\sum$ is normal at 0, it must satisfy the condition (β).*

Proof. Suppose that $\sum$ is normal at 0 and that $\sum$ does not satisfy the condition (β). Then there exist a point (x^0, y^0) near 0, satisfying $\delta(x^0) \neq 0, f(x^0, y^0) = 0$, and a system

$$z_\mu = z_\mu(x, y) = \sum_{\nu=0}^{\infty} c_\nu^{(\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p}},\ 1 \leqslant \mu \leqslant e,$$

such that $p>1$ and $c_1^{(\mu)}(x) \equiv 0$, $1 \leqslant \mu \leqslant e$.

Let $\nu_0, \nu_0 > 1$ be the first of the integers ν such that there is an index μ, $1 \leqslant \mu \leqslant e$, satisfying $c_\nu^{(\mu)}(x) \not\equiv 0$ and that ν is not divided by p. For simplicity, suppose that $c_{\nu_0}^{(1)}(x) \not\equiv 0$. Let τ, σ be the integers satisfying

$$\nu_0 = \tau p + \sigma,\ \tau \geqslant 0,\ 0 < \sigma < p.$$

And, for each μ, $1 \leqslant \mu \leqslant e$, let

$$G_\mu(x, y) = \sum_{\nu < \nu_0} c_\nu^{(\mu)}(x)(y-\varphi(x))^{\frac{\nu}{p}}.$$

Remark that the point $(x^0, y^0, z^0)(z_\mu^0 = z_\mu(x^0, y^0))$ is normal.

For example, suppose that $\tau > 0$, and consider the function

$$h = (y-\varphi(x))^{-\tau}(z_1 - G_1(x, y)),$$

which is holomorphic on $\sum$ in a neighborhood of (x^0, y^0, z^0). We see that h is not expressible by any holomorphic function $H(x, y, z)$ in the space (x, y, z), and we have a contradiction. In fact, suppose that such a function H exists. Let

$$H = \sum_{i_1, \cdots i_e = 0}^{\infty} b_{i_1 \cdots i_e}(x,y)(z_1 - G_1(x, y))^{i_1} \cdots (z_e - G_e(x, y))^{i_e}$$

2) See K. Oka: Sur les fonctions analytiques de plusieurs variables (Iwanami Shoten, Japan, 1961), especially p. 139, Lemme 1.

be the Taylor-series expansion of H. We have $b_{0\cdots 0}(x, \varphi(x))=0$, i.e., the quotient $b_{0\cdots 0}(x, y)/(y-\varphi(x))$ is bounded in a neighborhood of (x^0, y^0). Further, for each μ, $1\leqslant\mu\leqslant e$, $(z_\mu - G_\mu(x, y))/(y-\varphi(x))$ is also bounded there, when $(x, y, z)\in\sum$. Consequently, the quotient $H/(y-\varphi(x))$ is bounded on $\sum$ in a neighborhood of (x^0, y^0, z^0). But $h/(y-\varphi(x))$ is not bounded there, which shows that H cannot represent the function h.

In the case $\tau=0$, we arrive also at a contradiction, if we take the function

$$h=(y-\varphi(x))^{-\frac{1}{p}}(z_1-G_1(x, y)).$$

Proposition 5. *If $\sum$ satisfies the conditions (α) and (β), it is normal at* 0.

Proof. First suppose that $\sum$ is principal. Then $\sum$ is defined by the single equation $P=0$. Let (x^0, y^0, z_1^0) be a point on $\sum$, such that x^0 is near 0 and that $\delta(x^0)\neq 0$. If $f(x^0, y^0)\neq 0$, the point (x^0, y^0, z_1^0) is regular and then normal. If $f(x^0, y^0)=0$, there exists a unique Puiseux-series

$$z_1=\sum_{\nu=0}^{\infty} c_\nu(x)(y-\varphi(x))^{\frac{\nu}{p}},$$

which represents $\sum$ completely in a neighborhood of (x^0, y^0, z_1^0). If $p>1$, then, according to the condition (β), we have $c_1(x)\not\equiv 0$. In a neighborhood of (x^0, y^0, z_1^0), the analytic set $\sum_0$: $c_1(x)=0, y=\varphi(x)$, $z_1=c_0(x)$, includes completely the set of all non-normal points of $\sum$. By the lemma due to K. Oka we have referred to, we know that (x^0, y^0, z_1^0) is normal, because we have dim. $\sum=$dim. $\sum_0+2$.

Let (x^0, y^0, z_1^0) be a point of $\sum$ such that x^0 is near 0 and that $\delta(x^0)=0, f(x^0, y^0)\neq 0$. Then (x^0, y^0, z_1^0) is normal, for it is regular. Consequently, in a neighborhood of 0, the set $\sum_0'$: $P(x, y, z_1)=f(x, y)=\delta(x)=0$, includes the set of all non-normal points. By the same reason as above, we see that 0 is normal.

In the case where $\sum$ is not principal, doing a small linear change of the variables $z_1, \cdots, z_e$, and looking over the irreducible principal analytic set in the space (x, y, z_1), obtained by $\sum$, we are also able to arrive at the conclusion.

By the propositions 3, 4, 5, we have the

Theorem 1. *$\sum$ is normal, if and only if it satisfies the conditions (α) and (β).*

By the above theorem and the remark (b), we obtain the

Theorem 2. *The set of all non-normal points of $\sum$ is empty or purely $(d-1)$-dimensional, where $d=$*dim. $\sum$.