132. *On the Class of Paranormal Operators*

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1967)

Introduction. In this paper we discuss a class of non-normal operators. We call a bounded linear operator T on a Hilbert space H *paranormal* if $\|T^2x\| \geqq \|Tx\|^2$ for every unit vector x in H. In [4] this is named an operator of class (N). It is easily known that this class includes hyponormal operators and is included in the class of normaloid operators.*) We show these inclusion relations are proper and hence paranormal operators constitute a new class broader than hyponormal operators and narrower than normaloid operators.

I would like to express here my deep thanks to Professor Zirô Takeda for liberal use of his time and advice in the preparation of this paper.

1. Lemma 1. *Let T be a paranormal operator, then*

(1) $\|T^3x\| \geqq \|T^2x\| \cdot \|Tx\|$ *for every unit vector x in H.*

Proof. For a unit vector x in H, we may assume $Tx \neq 0$.

$$\|T^3x\| = \|Tx\| \cdot \left\| T^2 \frac{Tx}{\|Tx\|} \right\| \geqq \|Tx\| \cdot \left\| T \frac{Tx}{\|Tx\|} \right\|^2$$

$$= \frac{\|T^2x\|^2}{\|Tx\|} \geqq \frac{\|T^2x\| \cdot \|Tx\|^2}{\|Tx\|} = \|T^2x\| \cdot \|Tx\| \qquad \text{q.e.d.}$$

Lemma 2. *Let T be a paranormal operator, then*

(P_k) $$\|T^{k+1}x\|^2 \geqq \|T^kx\|^2 \cdot \|T^2x\|$$

for a positive integer $k \geqq 1$ and every unit vector x in H.

Proof. For the case $k=1$

$$\|T^2x\|^2 = \|T^2x\| \cdot \|T^2x\| \geqq \|Tx\|^2 \cdot \|T^2x\|$$

and (P_1) is clear. Now suppose that (P_k) is valid for k and we assume $\|Tx\| \neq 0$, then

$$\|T^{k+2}x\|^2 = \|Tx\|^2 \left\| \frac{T^{k+1}Tx}{\|Tx\|} \right\|^2 \geqq \|Tx\|^2 \left\| T^k \frac{Tx}{\|Tx\|} \right\|^2 \left\| T^2 \frac{Tx}{\|Tx\|} \right\|$$

$$= \|T^{k+1}x\|^2 \cdot \frac{\|T^3x\|}{\|Tx\|} \geqq \|T^{k+1}x\|^2 \cdot \|T^2x\|$$

by (1) of Lemma 1 and (P_k). So (P_{k+1}) is valid and the proof is complete by the mathematical induction. q.e.d.

Theorem 1. *If T is a paranormal operator, then T^n is paranormal for every integer $n \geqq 1$.*

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*) An operater T is said to be hyponormal if $T^*T \geqq TT^*$ and normaloid if $\|T^n\| = \|T\|^n$, (see definition 1).

Proof. It is sufficient to show that if T and T^k is paranormal, then T^{k+1} is paranormal too. We may assume $\|T^2x\|\neq 0$, then

$$\|T^{2(k+1)}x\|=\left\|T^{2k}\frac{T^2x}{\|T^2x\|}\right\|\cdot\|T^2x\|\geqq\left\|T^k\frac{T^2x}{\|T^2x\|}\right\|^2\cdot\|T^2x\|$$

$$=\frac{\|T^{k+2}x\|^2}{\|T^2x\|}\geqq\frac{\|T^{k+1}x\|^2\cdot\|T^2x\|}{\|T^2x\|}=\|T^{k+1}x\|^2$$

by (P_{k+1}) of Lemma 2. So T^{k+1} is paranormal. q.e.d.

Theorem 2. *There exists a paranormal operator which is not hyponormal. That is, the class of hyponormal operators is properly included in the class of paranormal operators.*

Proof. In [3] Halmos gives a hyponormal operator T such that T^2 is not hyponormal. By Theorem 1, this T^2 is paranormal. Hence we get an example of non-hyponormal, paranormal operator.

We discuss Halmos's example in next section.

In [2] Nakamoto and Horie have given a direct proof of the next theorem.

Theorem A. *A paranormal operator T is compact if and only if T^n is compact.*

In that paper the author has given an example of non-paranormal normaloid operator. For convenience sake, we show this example in next section again. Hence the class of paranormal operators is properly included in that of normaloid operators.

Generalizing the concept of normality, several authors have introduced classes of non-normal operators. Our new class occupies the place shown in the following schema and the inclusions are all proper.

$$\text{Normal}\subsetneqq\text{Quasi-normal}\subsetneqq\text{Subnormal}\subsetneqq\text{Hyponormal}$$
$$\subsetneqq\text{Paranormal}\subsetneqq\text{Normaloid}$$

The inclusion relations on the left hand side from hyponormal are well known in [8].

2. By several examples we indicate the inclusion relation between Classes of paranormal operators and convexoid operators.

Definition 1. An operator T is called to be normaloid if

$$\|T\|=\sup_{\|x\|=1}|(Tx,x)|.$$

It is known that T is normaloid if and only if the spectral radius is equal to $\|T\|$, or equivalently $\|T^n\|=\|T\|^n$ for all positive integers n ([1][3][5][7][8]).

Definition 2. An operator T is called to be convexoid if the closure of numerical range $\overline{W(T)}=\overline{\{(Tx,x)\colon \|x\|=1\}}$ equals to the convex hull of the spectrum $\sigma(T)$ of T.

It is known that there exists convexoid operators which are not normaloid and vice versa ([3]).

1) An example of non-convexoid, non-paranormal, normaloid operator ([2]).

Let T be an infinite matrix of the form

$$T=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & M & 0 & 0 & \cdots \\ 0 & 0 & M & 0 & \cdots \\ 0 & 0 & 0 & M & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \qquad \text{where } M=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then it is clear that T is normaloid ([2]), non-paranormal because

$$T^2=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and $\|T^n\|=\|T\|^n=1$. However the relation $\|T^2x\|\geqq\|Tx\|^2$ does not hold for the unit vectors $e_2=(0, 1, 0, 0, 0, \cdots)$, $e_4=(0, 0, 0, 1, 0, 0, \cdots)$ etc. T is non-convexoid. In fact $\overline{W(T)}$ is the closed convex set spanned by the disc $\{z: |z|\leqq 1/2\}$ and one point 1, $\sigma(T)=\{0\}\cup\{1\}$, so the convex hull of $\sigma(T)$ is the closed unit interval $[0, 1]$ and this unit interval is properly included in $\overline{W(T)}$.

Remark. Can we generalize Theorem A for normaloid operators ? This matrix gives a counterexample for this question, because T^2 is compact but T is not compact but normaloid.

2) An example of non-paranormal, convexoid, normaloid operator ([3]). Put $T=\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$ where $M=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as same as in example 1 and N be a normal operator whose spectrum is the closed unit disc $\bar{D}$. Then $\sigma(T)=\{0\}\cup\{\bar{D}\}=\bar{D}$, and $\overline{W(T)}=$the convex hull of $(W(M)\cup W(N))=\bar{D}$ and $\|T\|=1$. Hence T is convexoid and normaloid, but it is non-paranormal since $Te_1=e_2$, $T^2e_1=0$ for the unit vector $e_1=(1, 0, 0, \cdots), \cdots$.

3) An example (Halmos) of non-hyponormal, paranormal convexoid operator ([3]).

Let C and D be $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ respectively and give T, T^2 by the following infinite matrices respectively,

$$T=\begin{pmatrix} \ddots & & & & & \\ & C^{1/2} & 0 & 0 & & \\ & 0 & C^{1/2} & \boxed{0} & & \\ & 0 & 0 & C^{1/2} & & \\ & & & & D^{1/2} & \\ & & & & & D^{1/2} \\ & & & & & & \ddots \end{pmatrix}$$

$$T^2=\begin{pmatrix} \ddots & & & & & & \\ & C & & & & & \\ & & C & 0 & \boxed{0} & 0 & \\ & & 0 & C & 0 & 0 & \\ & & & 0 & D^{1/2}C^{1/2} & 0 & \\ & & & & 0 & D & \\ & & & & & & D \\ & & & & & & & \ddots \end{pmatrix}$$

($\square$ shows the place of the (0, 0) matrix element).

Clearly $D\geqq C$, but this does not imply $D^2\geqq C^2$. Basing on this fact we can ascertain that T is hyponormal but T^2 is not so. As well known every hyponormal operator is convexoid ([6][8]). We can confirm that this non-hyponormal, paranormal operator T^2 is also convexoid as follows.

D is a positive operator on a two dimensional space E. Its proper value are $(3+\sqrt{5})/2$ and $(3-\sqrt{5})/2$. Put $\mu=(3+\sqrt{5})/2$, clearly $1<\mu$ and $\|T\|=\sqrt{\mu}$, $\|T^2\|=\mu$.

Let $\varphi=(\varphi_1, \varphi_2)$ be the proper vector of D for the proper value μ and put $\psi=(\varphi_1, 0)$, $0=(0, 0)$. The matrix T is considered as an operator acting on the direct sum $\bigoplus_{n=-\infty}^{\infty} E_n$ where $E_n\cong E$. Take an arbitrary complex number λ such that $1<|\lambda|<\mu$ and put

$$\Phi=\left(\cdots 0, \frac{1}{\lambda^3}\psi, 0, \frac{1}{\lambda^2}\psi, 0, \frac{1}{\lambda}\psi, \boxed{0}, \varphi, 0, \frac{\lambda}{\mu}\varphi, 0, \frac{\lambda^2}{\mu^2}\varphi, 0, \frac{\lambda^3}{\mu^3}\varphi, 0 \cdots\right)$$

where each component is a vector in $E_n(-\infty<n<\infty)$ respectively and $\square$ shows the place of the 0-th coordinate. Then clearly Φ is a vector in $\bigoplus_{n=-\infty}^{\infty} E_n$ and by a simple calculation we can show that $T^{*2}\Phi=\lambda\Phi$.

This guarantees that every complex number λ such that $1<|\lambda|<\mu$ is in the spectrum $\sigma(T^2)$ and so the convex hull of the spectrum coincides with the disc $\{z: |z|\leqq\mu\}$. On the other hand since $\|T^2\|=\mu$, the numerical range of T^2 is contained in this disc. Hence T^2 is convexoid.

From this example naturally arises a question: *Is every paranormal operator is convexoid?* The answer is not certain for the present author.

Addendum. After we had written this manuscript, we found the following fundamental inequality. Let T be paranormal then the following inequality holds for every vector x, not necessarily for every unit vector x, by the homogeneity of T,

$$(*) \qquad \|T\| \geqq \cdots \geqq \frac{\|T^{n+1}x\|}{\|T^n x\|} \geqq \cdots$$
$$\geqq \frac{\|T^5x\|}{\|T^4x\|} \geqq \frac{\|T^4x\|}{\|T^3x\|} \geqq \frac{\|T^3x\|}{\|T^2x\|} \geqq \frac{\|T^2x\|}{\|T^1x\|} \geqq \frac{\|Tx\|}{\|x\|}.$$

From this inequality we get easily known properties of paranormal operators. For every unit vector x,

$$\frac{\|T^{2n}x\|}{\|T^n x\|} \geqq \frac{\|T^n x\|}{\|x\|}, \qquad \text{so } T^n \text{ is paranormal.}$$

T is normaloid, because we have

$$\frac{\|T^n x\|}{\|Tx\|} \geqq \left(\frac{\|Tx\|}{\|x\|}\right)^{n-1}$$

Lemma 2 of this paper follows from

$$\left(\frac{\|T^{k+1}x\|}{\|T^k x\|}\right)^2 \geqq \frac{\|T^2x\|}{\|Tx\|}\frac{\|Tx\|}{\|x\|} = \|T^2x\|.$$

If T is invertible, then the following inequality holds for every vector x,

$$(**) \qquad \frac{\|x\|}{\|T^{-1}x\|} \geqq \frac{\|T^{-1}x\|}{\|T^{-2}x\|} \geqq \frac{\|T^{-2}x\|}{\|T^{-3}x\|} \geqq \cdots$$
$$\geqq \frac{\|T^{-n+1}x\|}{\|T^{-n}x\|} \geqq \cdots \geqq \frac{1}{\|T^{-1}\|}$$

so $\|T^{-2}x\| \geqq \|T^{-1}x\|^2$, thus T^{-1} is paranormal.

Examining our preprint paper, Istratescu kindly informed us that he had also theorem 1 independently to us at almost same date and said that his proof was more computational.

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