128. On the Convergence Criterion of M. Izumi and S. Izumi

By Bun-Kong KWEE
Department of Mathematics, University of Malaya, Kuala Lumpur, Malaysia

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1. Introduction. Let $f(x)$ be a periodic function with period 2π and L-integrable over $[-\pi, \pi]$, and let

$$\phi(u)=\phi_x(u)=f(x+u)+f(x-u)-2f(x).$$

The following theorem on the convergence of Fourier series has been established by M. Izumi and S. Izumi [1]:

Theorem A. If

$$\int_0^t \phi(u)du=o(t) \quad (t\to 0), \tag{1}$$

and for some $\delta>0$, there is an $\alpha(0<\alpha<1)$ such that

$$\int_t^\delta |d(u^{-\alpha}\phi(u))|=o(t^{-\alpha}), \tag{2}$$

then the Fourier series of $f(x)$ converges to $f(x)$ at the point x.

This theorem is an extension of the following theorem of Tomic [2]:

Theorem B. If at the point x, $\phi(u)\to 0$ as $u\to 0$ and for $u\to 0$, $\phi(u)$ is slowly varying, then the Fourier series of $f(x)$ converges to $f(x)$ at the point x.

The aim of this paper is to discuss the relations between Izumi-Izumi's test and the following two tests:

Theorem C (Young). If (1) holds and

$$\int_0^t |d(u\phi(u))|=o(t), \tag{3}$$

then the Fourier series converges to $f(x)$.

Theorem D (Lebesgue). (1) and

$$\lim_{k\to\infty}\limsup_{t\to 0}\int_{kt}^{\pi}\frac{|\phi(u)-\phi(u+t)|}{u}du=0 \tag{4}$$

imply the convergence of the Fourier series of $f(x)$ at the point x.

We shall prove that Izumi-Izumi's test includes Young's but is included in Lebesgue's test.

2. The relation between M. Izumi - S. Izumi's and Young's test. We first prove that Izumi-Izumi's test includes Young's. It is enough to prove the following

Theorem 1. (3) *implies* (2).

Proof. Suppose that (3) holds. Then

$$\int_t^\delta |d(u^{-\alpha}\phi(u))| = \int_t^\delta |d(u\phi(u)\cdot u^{-1-\alpha})|$$
$$\leqslant \int_t^\delta \frac{1}{u^{1+\alpha}} d\int_t^u |d(v\phi(v))| + \int_t^\delta |u\phi(u)|\,|d(u^{-1-\alpha})|$$
$$= \left[\frac{1}{u^{1+\alpha}}\int_t^u |d(v\phi(v))|\right]_t^\delta + (1+\alpha)\int_t^\delta \frac{1}{u^{2+\alpha}}\int_t^u |d(v\phi(v))|\,du$$
$$+(1+\alpha)\int_t^\delta \frac{|\phi(u)|}{u^{1+\alpha}}du$$
$$= o(1) + o(t^{-\alpha}) + (1+\alpha)\int_t^\delta \frac{|\phi(u)|}{u^{1+\alpha}}du.$$

Since
$$t^\alpha \left|\frac{\phi(t)}{t^\alpha} - \frac{\phi(\delta)}{\delta^\alpha}\right| \leqslant t^\alpha \int_t^\delta |d(u^{-\alpha}\phi(u))| = o(1),$$
it follows that $\phi(t)$ is bounded in the interval $(0, \delta)$. Hence
$$\int_t^\delta |d(u^{-\alpha}\phi(u))| = o(t^{-\alpha}),$$
and this proves Theorem 1.

3. The relation between M. Izumi - S. Izumi's and Lebesgue's test. In this section we prove that Izumi-Izumi's test is included in Lebesgue's.

Theorem 2. *If* (2) *holds, then so does* (4).

Proof. Suppose that (2) holds. Then
$$\int_{kt}^\pi \frac{|\phi(u)-\phi(u+t)|}{u}du = \int_{kt}^{\delta-t} \frac{|\phi(u)-\phi(u+t)|}{u}du + \int_{\delta-t}^\pi \frac{|\phi(u)-\phi(u+t)|}{u}du$$
$$\leqslant \int_{kt}^{\delta-t}\left|\frac{\phi(u)}{u^\alpha} - \frac{\phi(u+t)}{(u+t)^\alpha}\right|\frac{du}{u^{1-\alpha}} + \int_{kt}^{\delta-t}\left(\frac{1}{u^\alpha} - \frac{1}{(u+t)^\alpha}\right)\frac{\phi(u+t)}{u^{1-\alpha}}du + o(1)$$
$$= I_1 + I_2 + o(1).$$

We have
$$I_1 \leqslant \int_{kt}^{\delta-t}\int_u^{u+t} |d(v^{-\alpha}\phi(v))|\frac{du}{u^{1-\alpha}}$$
$$\leqslant \int_{kt}^{\delta}\int_{v-t}^{v}\frac{du}{u^{1-\alpha}}|d(v^{-\alpha}\phi(v))|$$
$$\leqslant t\int_{kt}^\delta \frac{1}{(v-t)^{1-\alpha}}|d(v^{-\alpha}\phi(v))|$$
$$\leqslant \frac{t^\alpha}{(k-1)^{1-\alpha}}\int_{kt}^\delta |d(v^{-\alpha}\phi(v))|$$
$$= o\left(\frac{1}{k}\right).$$

Since $\phi(u+t)$ is bounded in the interval $(o, \delta-t)$,
$$I_2 = o\left(\int_{kt}^{\delta-t}\frac{(u+t)^\alpha - u^\alpha}{u(u+t)^\alpha}du\right)$$
$$= o\left(t\int_{kt}^\delta \frac{du}{u^\alpha(u+t)^\alpha}\right)$$
$$= o(1).$$

Hence (4) holds.

References

[1] M. Izumi and S. Izumi: A new convergence criterion of Fourier series. Proc. Japan Acad., **42**, 75-77 (1966).
[2] M. Tomić: A convergence criterion for Fourier series. Proc. Amer. Math. Soc., **15**, 612-617 (1964).