185. *A Note on the Generation of Nonlinear Semigroups in a Locally Convex Space*

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1. Let X be an (FM)-space, i.e., a Frechet space which is also a Montel space. For example, the space $H(\Omega)$ of holomorphic functions on a domain Ω in the complex plane, which is endowed with the topology of locally uniform convergence, the space $(\mathfrak{S})$ of rapidly decreasing functions on R^n and R^n are this space.

For a not necessarily linear operator A from X into itself, we introduce the following conditions:

(1) There exists a positive constant $\delta>0$ such that for each $h\in(0, \delta]$, the topological inverse mapping $(I-hA)^{-1}$ of the mapping $x\to x-hAx$ exists on X as a single valued operator.

(2) For any $T>0$, the family of operators $\{(I-hA)^{-n}\}$ is equicontinuous on X in $h\in(0, \delta]$ and n with $hn\in[0, T]$. (Put $(I-hA)^0=I$, the identity mapping.)

(3) For any $x\in D(A)$ and for any $T>0$, the set $\{A(I-hA)^{-n}x$: $h\in(0, \delta], hn\in[0, T]\}$ is bounded in X.

Definition 1. *A not necessarily linear operator A from X to itself is said to be of class $\mathfrak{A}$ if for this A all of the above conditions are satisfied.*

In the case that A is a densely defined closed linear operator, the well-known necessary and sufficient condition for A being the infinitesimal generator of an equicontinuous semigroup is rather stronger than the condition $A\in\mathfrak{A}$. We mention here some remarks on the abovementioned conditions:

(i) From (3) it follows that for any $x\in X$ the set $\{(I-hA)^{-n}x$: $h\in(0, \delta], hn\in[0, T]\}$ is bounded in X.

(ii) From (2) it follows that if $D(A)\ni x_n\to x$ and $Ax_n\to y$, then $x\in D(A)$ and $Ax=y$.

(iii) The following condition implies (2) and (3):

For any $x\in X$ and $T>0$ there exists a neighbourhood $U(x)$ of x such that for any continuous seminorm p there exists a continuous seminorm q which is independent of $h\in(0, \delta]$, n with $hn\in[0, T]$ and $z\in U(x)$, such that

$$p((I-hA)^{-n}x-(I-hA)^{-n}z)\leqq q(x-z), \qquad z\in U(x).$$

(iv) If A maps bounded sets in $D(A)$ into bounded sets, then

(3) can be replaced by the following

(3)′ For any $T>0$ there exists an $x_T \in X$ such that the set $\{(I-hA)^{-n}x_T: h \in (0, \delta], hn \in [0, T]\}$ is bounded in X.

Next we give the definitions of the semigroup of nonlinear operators and the infinitesimal generator.

Definition 2. *Let D be a closed subset in X. A one parameter family $\{T(t)\}_{t\geqq 0}$ of continuous mappings from D into itself is called to be a (nonlinear) semigroup on D, if the following conditions are satisfied:*

(4) $T(0)=I,\ T(t+s)=T(t)T(s)$ *on* D, $\qquad t, s\geqq 0.$

(5) *For each* $x \in D$, $T(t)x$ *is strongly continuous in* $t\geqq 0$.

And a semigroup $\{T(t)\}$ on D is called to be locally equicontinuous if for any $s>0$, $\{T(t)\}$ is equicontinuous on D in $t \in [0, s]$.

Definition 3. *We define the infinitesimal generator A_0 of a semigroup $\{T(t)\}_{t\geqq 0}$ mentioned above by*

$$A_0 x = \lim_{h \downarrow 0} h^{-1}(T(h)x - x)$$

whenever the limit exists.

Lately K. Kojima [2] gave the following result*): Let A be a continuous mapping on X into itself, for which (1), (2), and (3)′ are satisfied. Then it generates a nonlinear locally equicontinuous semigroup $\{T(t)\}$ on X in such a way that for each $x \in X$, $T(t)x$ is continuously differentiable at all $t\geqq 0$ and $T'(t)x = AT(t)x$, $t\geqq 0$.

In this paper we shall treat the generation of nonlinear semigroups for the mapping of class $\mathfrak{A}$ defined above. The main result is the following

Theorem. *Any mapping A of class $\mathfrak{A}$ generates a nonlinear locally equicontinuous semigroup $\{T(t)\}_{t\geqq 0}$ on $\overline{D(A)}$ in such a way that for each $x \in D(A)$,*

(6) $T(t)x \in D(A)$ *for all* $t\geqq 0$,

(7) $T(t)x$ *is continuously differentiable on* $t\geqq 0$ *and*

$$T'(t)x = AT(t)x, \qquad t\geqq 0.$$

Moreover let A_0 be its infinitesimal generator. If for some $h_0 \in (0, \delta]$, $I-h_0A_0$ is injective, i.e., for any pair of distinct elements x, y of $D(A)$ $(I-h_0A_0)x \neq (I-h_0A_0)y$, then $A_0 = A$.

2. Before proving the above theorem, we shall mention some important properties of (FM)-space, in the following

Proposition. *Let X be an (FM)-space. Then the following assertions are true:*

(a) *Any bounded set is sequentially compact.*

(b) *Any weakly convergent sequence is also strongly convergent*

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*) He obtains the result in the complete locally convex space such that it is separable and every bounded closed subset is sequentially compact.

to the same limit.

(c) *X is weakly sequentially complete as well as sequentially complete.*

(d) *Both X and the dual X^* are separable in the sense that there exists a countable dense subset $\{x_n\}\subset X$ (resp. $\{x_n^*\}\subset X^*$) and every element is the strong limit of a subsequence of $\{x_n\}$ (resp. $\{x_n^*\}$).*

The proof is omitted. See Köthe [4] and Edwards [5].

Now we give some notations used throughout this paper: We denote any strictly monotone increasing sequence of positive numbers tending to the infinity by $\{r_n\}$ and put $T(n, t)=(I-r_n^{-1}A)^{-[r_nt]}$ (where [ ] is the Gaussian blacket.) which are well-defined on X for all sufficiently large n.

Lemma. *Let $A\in\mathfrak{A}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\lim_{k\to\infty} T(n_k, t)x=T(t)x$ exists for each $x\in\overline{D(A)}$. And for any $s>0$, $\{T(t)\}$ is equicontinuous on $\overline{D(A)}$ in $t\in[0, s]$.*

Proof. For any $x\in D(A)$ and $t\geqq 0$, since $r_n^{-1}[r_nt]\leqq t$, it follows from (i) that $\{T(n, t)x\}_n$ is bounded in X, and so, from (a) of Proposition there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $\lim_{j\to\infty} T(n_j, t)x$ exists. And $D(A)$ is separable with respect to the relative topology, because X is a metric space. Let $\{t_j\}$ be the totality of rational numbers in $[0, \infty)$ and $\{x_i\}$ be a countable dense subset in $D(A)$. Then from the usual diagonal procedure we can find a subsequence $\{n_k\}$ of $\{n\}$ such that the limits $\lim_{k\to\infty} T(n_k, t_j)x_i$ exist for all i and j.

Let $[0, T]$ be a sufficiently large interval containing the t in question. From the simple calculation we have

$$r_n^{-1}\sum_{k=1}^{[r_nt]} A(I-r_n^{-1}A)^{-(k-1)}x=T(n, t)x-x+r_n^{-1}\{Ax-AT(n, t)x\}.$$

Since for any $x^*\in X^*$, $x^*AT(n, s)x$ is a step function on $[0, T]$, we get

$$(8)\quad x^*(T(n, t)x-x)=\int_0^t x^*AT(n, s)x\,ds + r_n^{-1}x^*\{AT(n, t)x-Ax\}+\int_{\frac{[r_nt]}{r_n}}^t x^*AT(n, s)x\,ds.$$

From (3) $\{AT(n, s)x\colon 0\leqq s\leqq T, n\}$ is bounded and so, $x^*AT(n, s)x$ is bounded measurable on $[0, T]$. Thus for any $t, t'\in[0, T]$ we have

$$(9)\quad |x^*T(n, t)x-x^*T(n,t')x| \leqq O(|t-t'|)+O(r_n^{-1})+O\left(\left|\frac{[r_nt]}{r_n}-t\right|\right)+O\left(\left|\frac{[r_nt']}{r_n}-t'\right|\right)$$

where O's depend only on $x\in D(A)$, $x^*\in X^*$ and $T>0$.

It follows from (9) that $\{T(n_k, t)x_i\}_k$ becomes a Cauchy sequence in the weak sense for each $t \geqq 0$. Thus from (c) and (b), $\lim_{k\to\infty} T(n_k, t)x_i$ exists for each $t \geqq 0$ and each i. Since $\{T(n, t)\}_n$ is equicontinuous, $\{T(n_k, t)x\}$ becomes a Cauchy sequence for each $t \geqq 0$ and each $x \in D(A)$ (and consequently for each $x \in \overline{D(A)}$). Therefore $\lim_{k\to\infty} T(n_k, t)x$ exists for each $t \geqq 0$ and $x \in \overline{D(A)}$. This convergence holds uniformly in t of every bounded interval [0, T].

We put $\lim_{k\to\infty} T(n_k, t)x = T(t)x$ on $\overline{D(A)}$. Since $\{T(n_k, t)\}_k$ is equicontinuous on a metric space $\overline{D(A)}$, $T(n_k, t)$ converges continuously to $T(t)$ on $\overline{D(A)}$ (see Rinow [6] p. 63). Take an arbitral $s > 0$. Then since $\{T(n, t)\}$ is equicontinuous on $\overline{D(A)}$ in n and $t \in [0, s]$ from (2), it follows that $\{T(t)\}$ is equicontinuous on $\overline{D(A)}$ in $t \in [0, s]$.

Proof of the theorem. Letting $k \to \infty$ in (9), we have

$$|x^*(T(t)x - T(t')x)| \leqq O(|t - t'|), \qquad t \geqq t' \geqq 0,\ x \in D(A),\ x^* \in X^*,$$

where 0 depends on x and x^* and $T > 0$. Since T is arbitral, $T(t)x$ is weakly continuous in $t \geqq 0$ and so, strongly continuous in $t \geqq 0$. Since $\{T(t)\}_{0 \leqq t \leqq T}$ is equicontinuous on $\overline{D(A)}$ from Lemma, $T(t)x$ is strongly continuous in $t \geqq 0$ for each $x \in \overline{D(A)}$.

From (3), $\{AT(n_k, t)x\}_k$ is bounded for each $x \in D(A)$ and $t \geqq 0$ and so, there exists a subsequence $AT(n_j, t)x$ converging to some element $y(t)$. Thus from (ii), $T(t)x \in D(A)$ for each $t \geqq 0$ and $\lim_{j\to\infty} AT(n_j, t)x = AT(t)x$. Here we may take the original sequence $\{n_k\}$ as this subsequence $\{n_j\}$. Since $AT(t)x$ is bounded on every finite interval $[0, T]$, again from (ii) it follows that $AT(t)x$ is strongly continuous in $t \geqq 0$. Therefore since X is sequentially complete, the Riemann integral $\int_0^t AT(s)x\,ds$ is defined in X for every $t \geqq 0$.

Thus letting $k \to \infty$ in (8), it follows from the dominated convergence theorem and the abovementioned that

$$x^*(T(t)x - x) = \int_0^t x^* AT(s)x\,ds = x^* \int_0^t AT(s)\,x\,ds$$

for each $t \geqq 0$, $x \in D(A)$ and each $x^* \in X^*$. Thus we have

$$T(t)x - x = \int_0^t AT(s)x\,ds, \qquad t \geqq 0,\ x \in D(A).$$

Therefore $T(t)x$ is strongly continuously differentiable and we have (7).

From the abovementioned, if $x \in D(A)$ then $T(t)x \in D(A)$ for all $t \geqq 0$. Thus from the continuity of each $T(t)$ on $\overline{D(A)}$, if $x \in \overline{D(A)}$ then $T(t)x \in \overline{D(A)}$ for all $t \geqq 0$. Thus $T(t)$ is a continuous mapping from $\overline{D(A)}$ into itself. Take any neighborhood V of 0 and $x \in D(A)$. Since $[r_n(s+t)] - [r_n s] - [r_n t] = \varepsilon$ is 0 or 1, we have $T(n, s+t)x - T(n, s)T(n, t)x \in V$ for all sufficiently large n. Since $\{T(n, t)\}_n$ is equicontinuous on $\overline{D(A)}$, the above estimate holds good for each

$x \in \overline{D(A)}$. Thus from the convergence $T(n_k, t) \rightarrow T(t)$ on $\overline{D(A)}$ and the equicontinuity of $\{T(n_k, t)\}_k$, it follows that for each $x \in \overline{D(A)}$ and for all sufficiently large n_k, $T(s+t)x - T(s)T(t)x \in 4V$. Thus $\{T(t)\}_{t \geqq 0}$ satisfies (4).

Finally, let A_0 be the infinitesimal generator of the above $\{T(t)\}$. Then clearly $A_0 \supseteqq A$. If $A_0 \supsetneqq A$, then it can be proved that the topological inverse mapping of $I - h_0 A_0$ must be multiple valued, which contradicts to the fact that $I - h_0 A_0$ is injective.

References

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