207. On Compactness in Ranked Spaces

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In this paper we will give a definition of compactness in the ranked space [1] and will prove some properties in respect of its compactness. We have used the same terminology as that introduced in the paper "On an Equivalence of Convergences in Ranked spaces" [3].

We say that the ranked space R satisfies the axiom (T_2) of separation, if and only if for any distinct points p and q of R there exist disjoint neighborhoods of p and of q respectively having certain ranks.

We say that the ranked space R satisfies the condition (M), if and only if for all points p of R the following condition is satisfied;

(M) if $V(p) \in \mathfrak{V}_\alpha$, $U(p) \in \mathfrak{V}_\beta$, and $\alpha \leq \beta$ then $V(p) \supseteq U(p)$.

Definition. *A subset A of the ranked space R is sequentially compact if and only if every sequence of A has a subsequence which is R-convergent to a point of A.*

Proposition 1. *Let R be the ranked space satisfying the axiom (T_2) of separation and the condition (M). If a sequence $\{p_\alpha\}$ of R is R-convergent, then $\{\lim_\alpha p_\alpha\}$ consists of only a point.*

Proof. Suppose $p, q \in \{\lim_\alpha p_\alpha\}$ and $p \neq q$. Since $p, q \in \{\lim_\alpha p_\alpha\}$, there exist a fundamental sequence $\{V_\alpha(p)\}$ of neighborhoods of p such that $p_\alpha \in V_\alpha(p)$ and a fundamental sequence $\{U_\alpha(q)\}$ of neighborhoods of q such that $p_\alpha \in U_\alpha(q)$. Hence, for all α

$$p_\alpha \in V_\alpha(p) \cap U_\alpha(q). \tag{1}$$

Since R satisfies the axiom (T_2), there exist a neighborhood $V(p)$ of p and a neighborhood $U(q)$ of q such that $V(p) \in \mathfrak{V}_\gamma$, $U(q) \in \mathfrak{V}_\delta$, and $V(p) \cap U(q) = \phi$.

By the condition (M), there exist $V_{\alpha_0}(p)$ and $U_{\alpha_0}(q)$ which are elements of $\{V_\alpha(p)\}$ and $\{U_\alpha(q)\}$ such that $V(p) \supseteq V_{\alpha_0}(p)$ and $U(q) \supseteq U_{\alpha_0}(q)$. Therefore, by (1) $p_{\alpha_0} \in V_{\alpha_0}(p) \cap U_{\alpha_0}(q) \subseteq V(p) \cap U(q)$, that is, $V(p) \cap U(q) \neq \phi$. This contradiction demonstrates that $\{\lim_\alpha p_\alpha\}$ consists of only a point.

Proposition 2. *Let R be the ranked space satisfying the*

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axiom (T_2) *of separation and the condition* (M). *If a subset* A *of* R *is sequentially compact and* $\{p_\alpha\}$ *is a* R*-convergent sequence of* A, *then* $\{\lim_\alpha p_\alpha\} \subseteq A$.

Proof. By the proposition 1, $\{p_\alpha\}$ is R-convergent to a single point p. Hence any subsequence $\{p_{\alpha_\beta}\}$ of $\{p_\alpha\}$ is R-convergent to p [1]. Also, by the proposition 1 $\{\lim_\beta p_{\alpha_\beta}\}$ consists of a point. On the other hand, since A is sequentially compact, $p \in A$.

Proposition 3. *Let* R *be the ranked space satisfying the axiom* (T_2) *of separation and the condition* (M). *If a subset* A *of* R *is sequentially compact then* A *is* m*-closed.*

Proof. Suppose that A is not m-closed. Since $R-A$ is not m-open, there exists some point x_0 belonging to $R-A$ such that $V(x_0) \cap A \neq \phi$ and $V(x_0) \in \mathfrak{B}_\gamma$ for all γ. On the other hand, since R is a ranked space, there exists a fundamental sequence $\{V_\alpha(x_0)\}$ of neighborhoods of x_0. Consequently, for every member $V_\alpha(x_0)$ of the fundamental sequence, there is a point p_α such that $p_\alpha \in V_\alpha(x_0) \cap A$. Hence $\{p_\alpha\}$ is a sequence of A and it is R-convergent to x_0 belonging to $R-A$. Since A is sequentially compact, by the proposition 2 x_0 is also contained in A. This contradiction demonstrates that A is m-closed.

Proposition 4. *Let* R *be a sequentially compact ranked space satisfying the condition* (M). *If a subset* A *of* R *is* m*-closed then* A *is sequentially compact.*

Proof. Let $\{p_\alpha\}$ be an arbitrary sequence of points in A. Since R is sequentially compact and $\{p_\alpha\}$ is a sequence of R, there is a point p of R and a subsequence $\{p_{\alpha_\beta}\}$ of $\{p_\alpha\}$ such that $p \in \{\lim_\beta p_{\alpha_\beta}\}$. Then, we have $p \in A$. Therefore, A is sequentially compact. In fact, if $p \notin A$ then $p \in R-A$. Since $R-A$ is m-open, by the proposition 3 of the previous paper [4] $\{p_{\alpha_\beta}\}$ is eventually in $R-A$. Hence $\{p_{\alpha_\beta}\}$ is not a sequence of A. This contradiction demonstrates that $p \in A$.

Proposition 5. *Let* f *be a continuous function* [2] *carrying a sequentially compact ranked space* X *onto a ranked space* Y. *Then* Y *is sequentially compact.*

Proof. Let $\{q_\alpha\}$ be an arbitrary sequence of points in Y. Since f is a mapping of X onto Y, for every q_α there is p_α belonging to X such that $f(p_\alpha) = q_\alpha$. Since X is sequentially compact, $\{p_\alpha\}$ has a subsequence $\{p_{\alpha_\beta}\}$ which is R-convergent to a point p of X. That is,

$$p \in \{\lim_\beta p_{\alpha_\beta}\}.$$

Since f is a continuous function, $f(p) \in \{\lim_\beta f(p_{\alpha_\beta})\}$. Hence Y is sequentially compact.

Remark 1. In the proposition 1 and 2, we assume that the ranked space satisfies the axiom (T_2) and the condition (M). If one of them fails then these propositions do not hold. The following examples show these facts.

Example 1. In the 2-dimensional Euclidean space R, we define a neighborhood $V_{n,l}(p_0)$ of a point $p_0=(x_0, y_0)$ with a rank n, as follows. Let $V_{n,l}(p_0)$ be the subset consisting of all points $p=(x, y)$ such that x and y satisfy the following inequalities:

(1) $-\infty<y<y_0$,

(2) $0\leq c^2(y-y_0)^2-(x-x_0)^2<c^2/n^2$,

(3) $0\geq c^2(x-x_0)-(y-y_0+l\sqrt{1+c^2})^2$,

where n is a natural number, l is a fixed positive number and c is a positive constant.

Then, R is a ranked space [1] and satisfies the condition (M), but R does not satisfy the axiom (T_2). In this ranked space R, a sequence $\{p_\alpha\}$ such as $p_\alpha=\left(0, \frac{1}{\alpha}\right)$ is R-convergent to the origin 0. However $\{\lim_\alpha p_\alpha\}$ does not consist of only a point. Therefore the proposition 1 does not hold. Moreover, let A be the subset that consists of $\{p_\alpha\}$ and the origin 0. A is sequentially compact. But, $\{\lim_\alpha p_\alpha\}\not\subseteq A$. Therefore the proposition 2 does not hold.

Example 2. When l is not fixed, R is still a ranked space and satisfies the axiom (T_2). But R does not satisfy the condition (M). In this case, the proposition 1 and 2 do not hold as in the example 1.

Remark 2. The ranked space R shown in the example 1 satisfies the axiom (T_0) [1] of separation and the condition (M), but the proposition 1 does not hold in R.

References

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