31. On the Representations of $SL(3, C)$. III

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In this part of the works we shall discuss unitary representations of G, including the supplementary series and the degenerate series.

1. It is already seen [1] that there exists the following invariant bilinear form on $\mathcal{D}_\chi \times \mathcal{D}_{\chi'}$ where $\chi=(l_1, m_1; \lambda_2, \mu_2)$ and $\chi'=(l_1, m_1; -l_1-\lambda_2, -m_1-\mu_2)$:

$$\int \delta^{(l_1, m_1)}(z_1')\varphi(z_1'z)\psi(z)dz_1'dz.$$

This form is degenerate, that is, if $\varphi \in \mathcal{E}_\chi^1$ or $\psi \in \mathcal{E}_{\chi'}^1$ we have $B(\varphi, \psi)=0$; moreover we obtain the following form on $\mathcal{E}_\chi^1 \times \mathcal{E}_{\chi'}^1$,:

$$B_1(\varphi, \psi)=(-1)^{p+q}p!(l_1-p-1)!q!(m_1-q-1)!$$

$$\times \int a_{pq}(z_2, z_3)b_{rs}(z_2, z_3)dz_2dz_3 \qquad (l_1-p-r-1=0 \text{ and } m_1-q-s-1=0),$$

$$=0 \qquad (l_1-p-r-1\neq 0 \text{ or } m_1-q-s-1\neq 0)$$

for $\varphi(z)=z_1^{(p,q)}a_{pq}(z_2, z_3)$ and $\psi(z)=z_1^{(r,s)}b_{rs}(z_2, z_3)$.

We remark that this form is equivalent to the non-degenerate form on $\mathcal{D}_{\chi^{s_1}}/\mathcal{F}^1_{\chi^{s_1}} \times \mathcal{D}_{\chi'^{s_1}}/\mathcal{F}^1_{\chi'^{s_1}}$:

$$\int z_1'^{(l_1-1, m_1-1)}\varphi(z'z)\psi(z)dz_1'dz.$$

In particular, if $l_1=1$ and $m_1=1$, the representation $\{T^\chi, \mathcal{E}_\chi^1\}$ is the so-called degenerate representation and bilinear form on $\mathcal{E}_\chi^1 \times \mathcal{E}_{\chi'}^1$ is clearly given by

$$\int a(z_2, z_3)b(z_2, z_3)dz_2dz_3.$$

2. Now we set $\langle\varphi, \psi\rangle=B(\varphi, \bar{\psi})$ for $\varphi, \psi \in \mathcal{D}_\chi$, where $\bar{\psi}$ is the complex conjugate of ψ and $\bar{\psi} \in \mathcal{D}_{\bar{\chi}}$, then $\langle\cdot, \cdot\rangle$ is an Hermitian form on $\mathcal{D}_\chi$. In case it exists and is positive definite, the representation $R(\chi)$ is unitary with respect to this scalar product.

(i) When $\chi\bar{\chi}(\delta)=1$, that is, $\lambda_1=(n_1+\sqrt{-1}\rho_1)/2$, $\mu_1=(-n_1+\sqrt{-1}\rho_1)/2$, $\lambda_2=(n_2+\sqrt{-1}\rho_2)/2$, $\mu_2=(-n_2+\sqrt{-1}\rho_2)/2$, where n_k are integers and ρ_k are real, then $\langle\varphi, \psi\rangle$ has the form $\int\varphi(z)\bar{\psi}(z)dz$ and is positive definite. Such representations are known as those of the principal series.

(ii) When $\chi\bar{\chi}^{s_1}(\delta)=1$, that is, $\lambda_1=\mu_1=\sigma$, $\lambda_2=-\sigma/2+(n-\sqrt{-1}\rho)/2$, $\mu_2=-\sigma/2+(-n-\sqrt{-1}\rho)/2$, where n is an integer, σ and ρ are real, then $\langle\varphi, \psi\rangle$ has the form $\int |z_1'|^{-2\sigma-2}\varphi(z_1'z)\overline{\psi(z)}dz_1'dz$.

When $\chi\overline{\chi}^{s_2}(\delta)=1$, that is, $\lambda_1=-\sigma/2+(-n+\sqrt{-1}\rho)/2$, $\mu_1=-\sigma/2+(n+\sqrt{-1}\rho)/2$, $\lambda_2=\mu_2=\sigma$, then $\langle\varphi, \psi\rangle$ has the form

$$\int |z_2'|^{-2\sigma-2}\varphi(z_2'z)\overline{\psi(z)}dz_2'dz.$$

When $\chi\overline{\chi}^{s_5}(\delta)=1$, that is, $\lambda_1=\sigma/2+(n-\sqrt{-1}\rho)/2$, $\mu_1=\sigma/2+(-n-\sqrt{-1}\rho)/2$, $\lambda_2=\sigma/2+(-n+\sqrt{-1}\rho)/2$, $\mu_2=\sigma/2+(n+\sqrt{-1}\rho)/2$, then $\langle\varphi, \psi\rangle$ has the form

$$\int (z_1'z_2'-z_3')^{(-\lambda_1-1,-\mu_1-1)}z_3'^{(-\lambda_2-1,-\mu_2-1)}\varphi(z'z)\overline{\psi}(z)dz'dz.$$

These three forms are positive definite if $-1<\sigma<1, \sigma\neq 0$ and corresponding unitary representations are mutually unitarily equivalent and are known as those of the supplementary series.

(iii) Let σ be a positive integer in the case (ii), and let $\lambda_1=\mu_1=m$, for instance. Then the form

$$\int \delta^{(m,m)}(z_1')\varphi(z_1'z)\overline{\psi(z)}dz'dz$$

is a positive definite form on $\mathcal{D}_\chi/\mathcal{E}_\chi^1$ and the representation $\{T^\chi, \mathcal{D}_\chi/\mathcal{E}_\chi^1\}$ is equivalent to $R(\chi')$, where $\lambda_1'=m, \mu_1'=-m, \lambda_2=-m/2+(n-\sqrt{-1}\rho)/2$, $\mu_2=m/2+(-n-\sqrt{-1}\rho)/2$, which is a representation of the principal series.

(iv) Let $\sigma=-1$ in the case (ii), then the bilinear forms are degenerate and they are positive definite either on $\mathcal{E}^1_{1,1;\lambda_2,\mu_2}(\lambda_2=1/2+(n-\sqrt{-1}\rho)/2$, $\mu_2=1/2+(-n-\sqrt{-1}\rho)/2$ or on $\mathcal{E}^2_{\lambda_1,\mu_1;1,1}(\lambda_1=1/2+(-n+\sqrt{-1}\rho)/2$, $\mu_1=1/2+(n+\sqrt{-1}\rho/2)$. In these forms the degenerate representations are unitary and mutually unitarily equivalent. They are known as those of the degenerate principal series.

(v) When $\chi=(p, q; q, p)$, p, and q being positive integers, the form

$$\int \delta^{(p,q)}(z_2')\delta^{(p+q,p+q)}(z_1')\delta^{(q,p)}(z_2'') \times \varphi(z_2'z_1'z_2''z)\overline{\psi}(z)dz_1'dz_2'dz_2''dz$$

is a positive definite form on the space $\mathcal{D}_\chi/\mathcal{A}_\chi$ and the representation $\{T^\chi, \mathcal{D}_\chi/\mathcal{A}_\chi\}$ is equivalent to $R(\chi')$ where $\chi'=(p, -p; q, -q)$. It is a representation of the principal series.

Theorem. *The representation $R(\chi)$ is unitary in the following three cases*:

(1) $\lambda_1=(n_1+\sqrt{-1}\rho_1)/2$, $\mu_1=(-n_1+\sqrt{-1}\rho_1)/2$, $\lambda_2=(n_2+\sqrt{-1}\rho_2)/2$, $\mu_2=(-n_2+\sqrt{-1}\rho_2)/2$, *where n_k are integers and ρ_k are real (principal series)*;

(2) $\lambda_1=\mu_1=\sigma$, $\lambda_2=-\sigma/2+(n-\sqrt{-1}\rho)/2$, $\mu_2=-\sigma/2+(-n-\sqrt{-1}\rho)/2$ *where n is an integer and σ and ρ are real and*

$-1<\sigma<1, \sigma\neq 0$ (*supplementary series*);

(3) $\lambda_1=\mu_1=-1, \lambda_2=1/2+(n-\sqrt{-1}\rho)/2, \mu_2=1/2+(-n-\sqrt{-1}\rho)/2$ (*degenerate principal series*).

Every irreducible unitary representation contained in $R(\chi)$ *is unitarily equivalent to that of above type.*

Reference

[1] M. Tsuchikawa: On the representations of $SL(3, \boldsymbol{C})$. I. Proc. Japan Acad., **43**, 852-855 (1967).