111. A Limit Theorem of a Pulse-Like Wave Form for a Markov Process

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Nagumo, Arimoto, and Yoshizawa [3] discussed the asymptotic behavior of the solution of the following equation, which describes an active pulse transmission line simulating an animal nerve axon,

$$\frac{\partial^3 u}{\partial t\,\partial x^2}=\frac{\partial^2 u}{\partial t^2}+\mu(1-u+\varepsilon u^2)\frac{\partial u}{\partial t}+u, \tag{1}$$

$$\mu>0,\quad \frac{3}{16}>\varepsilon>0,\quad x>0,\quad t>0,$$

with the boundary data $u(0, x)=0\ (x\geq 0)$, $\frac{\partial u}{\partial t}(0, x)=0\ (x\geq 0)$, and $u(t, 0)=\psi(t)\ (t\geq 0)$, $\psi(t)\equiv 0$ for $t\geq t_0$. The equation (1) may be written as a system of equations

$$(1')\qquad \begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}-\mu\left(u-\dfrac{1}{2}u^2+\dfrac{\varepsilon}{3}u^3\right)-w \\ \dfrac{\partial w}{\partial t}=u. \end{cases}$$

They showed experimentally that the solution is a specific pulse-like wave form such that, when t increases, smaller signals are amplified, larger ones are attenuated, narrower ones are widen and those which are wider are shrunk, all approaching a specific wave form; and there is a threshold value to the signal height, and signals below the threshold (or noise) are eliminated when $t\to\infty$.

A probabilistic model for the Nagumo *et al.*'s equation was given in terms of a branching Markov process with age and sign in [2].[1] Since such a limiting property stated above is new in the theory of Markov processes, it is an attractive problem to discuss the limit theorem[2] in connection with the probability theory.[3] The objective

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1) For branching Markov processes with age and sign cf. also [5].

2) Yamaguchi proved some limit theorem for (1′) in [6].

3) H. P. McKean obtained some results of the problem for FitzHugh's equation, a version of (1′),

$$\begin{cases} \dfrac{1}{c}\dfrac{du}{dt}=\left(u-\dfrac{u^3}{3}\right)+w, \\ c\dfrac{dw}{dt}=a-u, \quad \text{(private communication).} \end{cases}$$

of this note is to indicate the fact that the same limit theorem as for an animal nerve axon holds for a simplified probabilistic model.

Consider the following motion: Suppose a particle moving uniformly on a line with speed ε, getting older with an exponential holding time $\exp\left(-\frac{kb}{\alpha\beta}t\right)$, where $k>0, 0<\alpha<\beta$, and $b=1+\alpha+\beta+\alpha\beta$; that is, we are considering the product (x_t, k_t) of the uniform motion x_t and a Poisson process k_t with the rate $\frac{kb}{\alpha\beta}$ which is independent of x_t. Suppose there are two worlds marked by $\{0\}$ and $\{1\}$ respectively. A particle with age (x_t, k_t) moving uniformly in the world $\{0\}$ makes transition, with the exponential holding time $\exp\left(-\frac{kb}{\alpha\beta}t\right)$, to a particle of the same age in the world $\{1\}$ with probability $|q_1|$, to two particles in the world $\{0\}$, one of which is of the same age and another one of age zero (a baby), with probability $|q_2|$, or to three particles in the world $\{1\}$, one is of the same age and the other two of age zero, with probability $|q_3|$, where

$$q_1=-\frac{\alpha\beta}{b},\ q_2=\frac{\alpha\beta}{b},\ q_3=-\frac{1}{b},\quad \text{and}\quad b=1+\alpha+\beta+\alpha\beta.$$

Then they move and get older independently each other until the next transition occurs. Suppose there are n-particles in the world $\{0\}$ ($\{1\}$) at the moment (suppose these n-particles constitute a family), then only one member of the family splits into i) one, ii) two or iii) three particles in the same mechanism as above with probability $|q_1|, |q_2|$ or $|q_3|$, respectively, while the family (which consists of $n, n+1$ or $n+2$ particles according to the splitting) immigrates to another world ($\{1\}$ or $\{0\}$ resp.) if the case i) or ii) has occurred in the splitting, and stays in the same world ($\{0\}$ or $\{1\}$ resp.) if the case ii) has occurred. The state of the motion of particles described above is, therefore, specified by a point $((x^1, k^1), (x^2, k^2), \cdots, (x^n, k^n))\times\{j\}, n=1, 2, 3, \cdots$, $x^i\in(-\infty, \infty)$, $k^i\in\{0, 1, 2, 3, \cdots\}$ and $j\in\{0, 1\}$. ($((x^1, k^1), \cdots, (x^n, k^n))$ represents members of a family and j stands for the world they live in.) Thus the process $\boldsymbol{Z}_t=((x_t^1, k_t^1), (x_t^2, k_t^2), \cdots, (x_t^{\xi(t)}, k_t^{\xi(t)}))\times\{j_t\}$, where $\xi(t)$ is the number of particles in a family at t, is a (strong) Markov process[4] (cf. [2]), which will be called a branching uniform motion with age and sign. Set, for a non-negative continuous function f on $(-\infty, \infty)$ with compact support,

$$(2)\qquad \widetilde{f\cdot 2}(x, k, j)=(-1)^j\prod_{i=1}^{n} f(x^i)2^{|k|},$$

when $\boldsymbol{x}=(x^1, x^2, \cdots, x^n)$, $\boldsymbol{k}=(k^1, k^2, \cdots, k^n)$, where $|\boldsymbol{k}|=\sum_{i=1}^{n}k^i$. Then de-

4) A sample path of $\boldsymbol{Z}_t$ can be understood as a family history.

fine $u(t, x)$ by

$$u(t, x)=\boldsymbol{E}_{(x,0,0)}[\widetilde{f\cdot 2}(\boldsymbol{Z}_t)], \tag{3}$$

where $\boldsymbol{E}_{(t,0,0)}$ denotes the expectation under the condition that a particle of age zero exists at x in the world $\{0\}$ when $t=0$. Probabilistic meaning of $u(t, x)$ is given, roughly speaking, as the mean value of the total charge of particles in a family, if we understand that each particle carries charge $+1$ or -1 according to the world $\{0\}$ or $\{1\}$ where the family lives in. Now we have

Theorem. *Define $u(t, x)$ by (3) for a non-negative continuous function f on $(-\infty, \infty)$ with compact support. Put $\Gamma=\{a\,;\, f(a)>\alpha\}$. Then, $u(t, x)$ converges asymptotically to $\beta\chi_\Gamma(x+\varepsilon t)$[5] when t tends to infinity. More precisely,*

i) *when $x+\varepsilon t\in\Gamma_1=\{a\,;\,\beta<f(a)\}$, $u(t, x)$ decreases to β $(t\to\infty)$,*

ii) *when $x+\varepsilon t\in\Gamma_2=\{a\,;\,\alpha<f(a)<\beta\}$, $u(t, x)$ increases to $\beta(t\to\infty)$,*

and

iii) *when $x+\varepsilon t\in\Gamma'=\{a\,;\,0<f(a)<\alpha\}$, $u(t, x)$ decreases to $0(t\to\infty)$.*

Proof. It is shown that $u(t, x)$ defined by (3) is the unique solution of the following semi-linear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t}=\varepsilon\dfrac{\partial u}{\partial x}+\dfrac{kb}{\alpha\beta}\left(-\dfrac{1}{b}u^3+\dfrac{\alpha+\beta}{b}u^2-\dfrac{\alpha\beta}{b}u\right) \\ u(0, x)=f(x), \end{cases} \tag{4}$$

(cf. [2], [3]). Since the characteristic line of (4) is $x+\varepsilon t=a$, setting $v(t, a)=u(t, a-\varepsilon t)$, we have

$$\begin{cases} \dfrac{dv}{dt}=-\dfrac{k}{\alpha\beta}v(v-\alpha)(v-\beta) \\ v(0, a)=f(a). \end{cases} \tag{5}$$

The solution of (5) is given by

$$\frac{|v|\,|v-\beta|^{\frac{\alpha}{\beta-\alpha}}}{|v-\alpha|^{\frac{\alpha}{\beta-\alpha}}}=A(a)e^{-kt},\quad A(a)=\frac{f(a)|\,f(a)-\beta|^{\frac{\alpha}{\beta-\alpha}}}{|\,f(a)-\alpha|^{\frac{\beta}{\beta-\alpha}}}. \tag{6}$$

It is easy to see that i) $dv/dt<0$ when $\beta<v$, ii) $dv/dt>0$ when $\alpha<v<\beta$, and iii) $dv/dt<0$ when $0<v<\alpha$. Thus 0 and β are stable critical points and α is an unstable critical point.[6] Since $u(t, x)=v(t, x+\varepsilon t)$, $u(t, x)$ behaves like $v(t, a)$ on the characteristic line $x+\varepsilon t=a$. Therefore, i) when $x+\varepsilon t\in\Gamma_1$, $u(t, x)\uparrow\beta$ $(t\to\infty)$, ii) when $x+\varepsilon t\in\Gamma_2$, $u(t, x)\downarrow\beta$ $(t\to\infty)$, and iii) $u(t, x)\downarrow 0$ $(t\to\infty)$ when $x+\varepsilon t\in\Gamma'$.

5) χ_Γ is the indicator of the set Γ.

6) Cf., for example, Petrovski [4].

References

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