108. On the Hölder Continuity of Stationary Gaussian Processes

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Let $X=\{X(t)\,;\,-\infty<t<\infty\}$ be a real, separable and stochastically continuous stationary Gaussian process with mean zero and with the covariance function $\rho(t)=E(X(t+s)X(s))$. Without loss of generality, we may assume $\rho(0)=1$. The continuity of path functions of X has been studied by many authors and further, under the rather strong condition on $\sigma^2(t)=E((X(t+s)-X(s))^2)=2(1-\rho(t))$, the Hölder continuity of $X(t, w)$[1)] was discussed by Yu. K. Belayev in his [1], among others. Our purpose in this paper is to give the final result about the Hölder continuity of $X(t, w)$ under the similar conditions to Belayev's one. In the case of Brownian motion with d-dimensional parameter, the same problem was solved by T. Sirao [3]. We will state our result in the form corresponding to the Brownian case. After the Brownian case, we first introduce the notions of the upper class and lower class for $\{X(t)\,;\,0\leqq t\leqq 1\}$. If there exists a positive number δ such that $|t-s|\leqq\delta$ $(0\leqq t, s\leqq 1)$ implies

$$|f(t)-f(s)|\leqq g(|t-s|),$$

then we say that $f(t)$ satisfies Lipschitz's condition relative to $g(t)$. Let $\varphi(t)$ be a positive, non-decreasing and continuous function defined for large t's. If almost all sample functions $X(t, w)$ satisfy (do not satisfy) Lipschitz's condition relative to $g(t)=\sigma(t)\varphi(1/t)$, then we say that $\varphi(t)$ belongs to the upper (lower) class with respect to the uniform continuity of $\{X(t)\,;\,0\leqq t\leqq 1\}$ and denote it by $\varphi\in\mathcal{U}^u(\mathcal{L}^u)$.

Next, we consider following Condition (A) consisting in (A. 1) and (A. 2).

(A. 1) There exist constants $0<\alpha<2$, $-\infty<\beta<\infty$, and $\delta>0$ such that for any h in $(0, \delta)$

$$C_1\frac{h^\alpha}{|\log h|^\beta}\leqq\sigma^2(h)\leqq C_2\frac{h^\alpha}{|\log h|^\beta},\quad 0<C_1<C_2<\infty.$$

(A. 2) $\sigma^2(h)$ is concave in $(0, \delta)$ if either one of $0<\alpha<1$, $-\infty<\beta<\infty$ or $\alpha=1$, $\beta\leqq 0$ holds and $\sigma^2(h)$ is convex in $(0, \delta)$ if either one of $\alpha>1$, $-\infty<\beta<\infty$ or $\alpha=1$, $\beta\geqq 0$ holds, where α, β, γ are constants mentioned in (A. 1).

1) w denotes a probability parameter.

Then we have

Theorem 1. *Let Condition* (A) *be satisfied and* $\varphi(t)$ *be a positive, non-decreasing continuous function defined for large t's. If, for some* $a>0$,

$$\int_a^\infty \varphi(t)^{\frac{4}{\alpha}-1} \exp\left(-\frac{1}{2}\varphi^2(t)\right) dt < \infty,$$

then the function $\varphi(t)$ *belongs to* $\mathcal{U}^u$.

We can easily deduce the following

Corollary 1.1. *Under Condition* (A), *we have for* $\varepsilon>0$

$$\left\{2\log t + \left(\frac{4}{\alpha}+1\right)\log_{(2)} t + 2\log_{(3)} t + \cdots + 2\log_{(n-1)} t + (2+\varepsilon)\log_{(n)} t\right\}^{\frac{1}{2}} \in \mathcal{U}^u,$$

where $\log_{(n)} t$ *denotes the n-time iterated logarithm.*

About the lower class, we have

Theorem 2. *Suppose that* $0<\alpha<1$ or $\alpha=1$, $\beta\leqq 0$ *and Condition* (A) *is satisfied. If* $\varphi(t)$ *is a positive, non-decreasing and continuous function defined for large t's and if for a positive a, the integral*

$$\int_a^\infty \varphi(t)^{\frac{4}{\alpha}-1} \exp\left(\frac{1}{2}\varphi^2(t)\right) dt = \infty,$$

then $\varphi(t)$ *belongs to* $\mathcal{L}^u$.

Also, under the assumptions as in Theorem 2, we can immediately obtain

Corollary 2.1. *If* $\varepsilon\geqq 0$,

$$\varphi(t) = \left\{2\log t + \left(\frac{4}{\alpha}+1\right)\log_{(2)} t + 2\log_{(3)} t + \cdots + 2\log_{(n-1)} t + (2-\varepsilon)\log_{(n)} t\right\}^{\frac{1}{2}} \in \mathcal{L}^u.$$

Combining Corollary 1.1 *with Corollary* 2.1, *we have*

Corollary 2.2. *Under the same assumption as in Theorem* 2,

$$P\left(\limsup_{h\to 0}\left\{\frac{|\xi(t)-\xi(s)|}{\sigma(t-s)\{2|\log|t-s||\}^{\frac{1}{2}}};\ 0\leqq t, s\leqq 1, 0<|t-s|\leqq h\right\}=1\right)=1.$$

Theorems 1 and 2 will be proved in the way similar to that of [3].

Remark 1. Corollary 2.2 is a refinement of Belayev's ones. Using our notations, his result is stated as follows: Under the assumptions that $\sigma^2(h)$ is concave and Condition (A. 1) holds for $\beta=1$, the function

$$\varphi_c(t) = \frac{c}{\sigma\left(\frac{1}{t}\right) t^{\frac{\alpha}{2}}} \left(\leqq \frac{c}{\sqrt{C_1}}(\log t)^{\frac{1}{2}}\right)$$

belongs to $\mathcal{L}^u$ if $c<\sqrt{2C_1}$ and belongs to $\mathcal{U}^u$ if $c>2\sqrt{C_2}\cdot 2^{\alpha}/2^{\alpha/2}-1(>\sqrt{2C_2})$. In the latter case, we have

$$\varphi_c(t) \geqq \frac{c}{\sqrt{C_2}}(\log t)^{\frac{1}{2}} > (2\log t)^{\frac{1}{2}}.$$

Remark 2. An interesting fact for us is that according to Corollaries 1.1 and 2.1, $\sigma(t)$ does not give any influence with exception of the second term for the criterion whether the function $\varphi(t)$ belongs to $\mathcal{U}^u$ or not, if $\varphi(t)$ is expressed in the form of sum of iterated logarithms. We are not sure if it is true or not when we exchange Condition (A. 1) for a more weak condition.

Remark 3. Condition (A) excludes all the case for $\alpha=0$ which contain the critical case where almost all sample functions are continuous or not.[2)]

References

[1] Yu. K. Belayev: Continuity and Hölder's conditions for sample functions of stationary Gaussian processes. Proc. Fourth Berkeley Symp. Math. Stat. Prob., **2**, 23–34 (1961).
[2] M. X. Fernique: Continuité des processus Gaussiens. Compt. Rend. Acad. Sci. Paris, **258**, 6058–6060 (1964).
[3] T. Sirao: On the continuity of Brownian motion with a multidimensional parameter. Nagoya Math. Jour., **16**, 135–156 (1960).

2) cf. M. X. Fernique [2].