105. On Generalized Commuting Properties of Metric Automorphisms. I

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(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1968)

Adler [1] has proved that the generalized commuting order of a totally ergodic automorphism on a compact metric abelian group is two. We shall prove that the generalized commuting order of a totally ergodic metric automorphism on the measure algebra associated with a finite measure space is two. The study in this paper depends on Adler's idea in [1].

The author wishes to express his thanks to Professor S. Tsurumi for his variable suggestions.

Let (X, Σ, m) be a finite measure space where X is a set of elements, Σ a σ-field of measurable subsets of X, and m a finite measure on Σ. A measure algebra associated with the measure space (X, Σ, m) is the Boolean algebra formed by identifying sets in Σ whose symmetric difference has measure zero. An automorphism of the measure algebra is called a *metric automorphism*. Let G be the group of all metric automorphisms on the measure algebra with the identity I. $C_n(T)$, $n=1, 2, \cdots$ of subfamilies of G associated with a metric automorphism T are defined inductively as follows:

$$C_0(T)=\{I\},$$
$$C_n(T)=\{S \in G : STS^{-1}T^{-1} \in C_{n-1}(T)\}, \quad n=1, 2, \cdots.$$

If there exists an integer N such that $C_N(T)=C_{N+1}(T)$ then $C_n(T)=C_{n+1}(T)$ for all $n \geqslant N$ and in this case we define $N(T)=\min\{N : C_N(T)=C_{N+1}(T)\}$ and otherwise $N(T)=\infty$. $N(T)$ is called the *generalized commuting order* of T. Let $L^2(X)$ be the Hilbert space of complex-valued square integrable functions defined on (X, Σ, m) and $L^\infty(X)$ the Banach space of complex-valued m essentially bounded functions defined on (X, Σ, m). A metric automorphism T is said to have *discrete spectrum* if there is a basis $\mathbf{0}$ of $L^2(X)$ each term of which is a normalized proper function of the linear isometry V_T induced by T. Clearly $\mathbf{0}$ includes the circle group K in the complex plane. If T is ergodic then it turns out that $|f|=1$ a.e. for each $f \in \mathbf{0}$ and that $\mathbf{0}=\mathbf{0}(T)\times K$ where $\mathbf{0}(T)$ is a subgroup of $\mathbf{0}$ isomorphic to the factor group $\mathbf{0}/K$ [4]. If f is a proper function of T and α its proper value, then we denote by $\alpha_T(f)$ the proper value α. T is said to be *totally ergodic* if T^n is ergodic for every positive integer n.

Proposition 1. *If a metric automorphism T is ergodic and has discrete spectrum, then $S \in C_1(T)$ if and only if every element of $\mathbf{0}(T)$ is a proper function of S.*

Proof. We prove first the "if" part. For each $f \in \mathbf{0}(T)$, we have

$$V_T(V_S f) = V_S V_T f = \alpha_T(f)(V_S f) \text{ a.e.}, \qquad V_T f = \alpha_T(f) f \text{ a.e.}$$

By proper value theorem we have $V_S f = \alpha_S(f) f$ a.e., where $\alpha_S(f)$ is a constant of absolute value one which depends only on the proper function f. Hence every $f \in \mathbf{0}(T)$ is a proper function of S.

We prove next the "only if" part. Since each function $f \in \mathbf{0}(T)$ is a proper function of T and also of S, we have

$$V_T V_S f = V_S V_T f \text{ a.e.}$$

for each $f \in \mathbf{0}(T)$. This yields $TS = ST$, namely $S \in C_1(T)$.

If A is a subalgebra of $L^\infty(X)$ such that A is closed under complex conjugation and $L^2(X) = \overline{\text{span}\, A}$, and if V is a linear isometry of $L^2(X)$ onto itself satisfying $\|Vp\|_\infty = \|p\|_\infty$ for each $p \in A$, then $VL^\infty(X) = L^\infty(X)$ [2].

Proposition 2. *Let T be an ergodic metric automorphism with discrete spectrum and let $S_2 \in C_2(T)$. Then there exist metric automorphism W and S such that W has each element of $\mathbf{0}(T)$ as a proper function and the linear isometry V_S induced by S maps $\mathbf{0}(T)$ onto itself and $S_2 = SW$.*

Proof. There exists $S_1 \in C_1(T)$ such that $S_2 T S_2^{-1} T^{-1} = S_1$. Hence $V_{S_2} V_T = V_S V_{S_2}$ where $S = S_1 T$. For each $f \in \mathbf{0}(T)$

$$V_T(V_{S_2}^{-1} f) = \alpha_S(f)(V_{S_2}^{-1} f) \text{ a.e.}, \qquad V_T f = \alpha_T(f) f \text{ a.e.}$$

Since proper functions associated with different proper values are orthogonal in $L^2(X)$ and $\mathbf{0}(T)$ is an orthonormal base of $L^2(X)$, there exists uniquely $g \in \mathbf{0}(T)$ such that $\alpha_T(g) = \alpha_S(f)$ for each $f \in \mathbf{0}(T)$. Now we define $Uf = g$. We prove that U is a one-to-one mapping of $\mathbf{0}(T)$ onto itself. By $V_{S_2} V_T = V_S V_{S_2}$ we have

$$V_T(V_{S_2}^{-1} f) = \alpha_S(f)(V_{S_2}^{-1} f) \text{ a.e.}, \qquad V_T(Uf) = \alpha_T(g)(Uf) \text{ a.e. } (Uf = g).$$

For each $f \in \mathbf{0}(T)$ there exists a complex number $\beta(f)$ of the absolute value one such that $V_{S_2}^{-1} f = \beta(f) Uf$ a.e. Since $V_{S_2}^{-1} \mathbf{0}(T)$ is an orthonormal base of $L^2(X)$, so is a set $\{Uf : f \in \mathbf{0}(T)\}$. Thus we can conclude that U maps $\mathbf{0}(T)$ onto itself. It remains to show that U is one-to-one. Suppose $Uf_1 = Uf_2$ a.e. for $f_1, f_2 \in \mathbf{0}(T)$ with $f_1 \neq f_2$ a.e. Then we have

$$V_{S_2}^{-1} f_1 = \beta(f_1) U f_1 \text{ a.e.}, \qquad V_{S_2}^{-1} f_2 = \beta(f_2) U f_2 = \beta(f_2) U f_1 \text{ a.e.}$$

Thus

$$V_{S_2}^{-1}(\gamma f_1) = V_{S_2}^{-1} f_2 \text{ a.e.}$$

where γ is a complex number such that $\gamma \beta(f_1) = \beta(f_2)$. Since V_{S_2} is one-to-one, we obtain $\gamma f_1 = f_2$ a.e. This is a contradiction.

Next, we put

$$V(\sum_{i=1}^{n} r_i f_i)=\sum_{i=1}^{n} r_i U f_i \quad (f_i \in \mathbf{0}(T)).$$

Then $||V(\sum_{i=1}^{n} r_i f_i)||_2=||\sum_{i=1}^{n} r_i f_i||_2$. Hence V is an isometry which can be extended uniquely to that of $L^2(X)$ onto itself. Furthermore, we put $Rf=\beta(U^{-1}f)f$ for each $f \in \mathbf{0}(T)$. Then R is a one-to-one mapping of $\mathbf{0}(T)$ onto a set $\{\beta(U^{-1}f)f : f \in \mathbf{0}(T)\}$ and has a unique continuous extension V' such that $V'f=Rf$ a.e. for each $f \in \mathbf{0}(T)$. Denote by $A(T)$ the space of polynomials $\sum_{i=1}^{n} r_i f_i (f_i \in \mathbf{0}(T))$. Then $A(T)$ is an algebra containing all polynomials and also their complex conjugations. Since $\mathbf{0}(T)$ is an abelian group the operators V and V' are multiplicative on $A(T)$. Thus, for each $p \in A(T)$, $||(Vp)^n||_2 \leqslant m(X)^{1/2}||p||_\infty^n$ where $||p||_\infty=$ess. sup $|p|$. Consequently $||Vp||_\infty \leqslant ||p||_\infty$. Similarly we have $||V^{-1}p||_\infty \leqslant ||p||_\infty$. Therefore $||Vp||_\infty=||p||_\infty$ and hence $VL^\infty(X)=L^\infty(X)$. Since, for $f, g \in L^\infty(X)$, we can choose $\{p_n\}, \{q_n\} \subset A(T)$ so that $||p_n-f||_2 \to 0$ as $n \to \infty$ and $||q_n-g||_2 \to 0$ as $n \to \infty$, $V(fg)=VfVg$ a.e. By multiplication theorem, there exists a metric automorphism S^{-1} which induces V. The same argument leads that for V' there exists a metric automorphism W^{-1} which induces V'. Consequently we have

$$V_{S_2}=V_S V_W, \qquad \text{namely } S_2=SW.$$

Proposition 3. *Let T be a totally ergodic metric automorphism with discrete spectrum, S a metric automorphism whose induces isometry V_S maps $\mathbf{0}(T)$ onto itself, and W a metric automorphism which has every element in $\mathbf{0}(T)$ as a proper function of W. If there exists a metric automorphism S' such that $S'TS'^{-1}T^{-1}=SW$, then S is an identity.*

Proof. Suppose that $f \in \mathbf{0}(T)$ and $Q=SW$, and consider the Fourier expansion of $V_{S'}f$, $V_{S'}f=\sum_i \langle V_{S'}f, f_i\rangle f_i$ a.e. $(f_i \in \mathbf{0}(T))$. Then, by $V_{S'}V_T=(V_Q V_T)V_{S'}$, we have

$$\sum_i \alpha_T(f)\langle V_{S'}f, f_i\rangle f_i=\sum_i \alpha_T(f_i)\alpha_W(f_i)\langle V_{S'}f, f_i\rangle V_S f_i \text{ a.e.}$$

and so

$$\alpha_T(f)\langle V_{S'}f, V_S f_i\rangle=\alpha_T(f_i)\alpha_W(f_i)\langle V_{S'}f, f_i\rangle$$

for each i. If f_i has an infinite orbit under V_S then infinitely many of the coefficients must have the same absolute value $|\langle V_{S'}f, f_i\rangle|$. On the other hand, the coefficients are square summable and so $\langle V_{S'}f, f_i\rangle=0$. Thus each f_i with non-zero coefficient has a finite orbit under V_S. $V_{S'}\mathbf{0}(T)$ is an orthonormal base of $L^2(X)$, and for each $f \in \mathbf{0}(T)$, $V_{S'}f$ can be expanded in terms of elements of $\mathbf{0}(T)$ which are periodic under V_S. Therefore the set of proper functions in $\mathbf{0}(T)$ which are periodic under V_S is also an orthonormal base of $L^2(X)$. We have shown that every $f \in \mathbf{0}(T)$ must have a finite orbit under V_S. Suppose now $V_S^n f=f$ a.e. for some $f \in \mathbf{0}(T)$. Since

$$(V_Q V_T)^n f = \alpha_T(f)\alpha_W(f)\alpha_T(V_S f)\alpha_W(V_S f) \cdots \alpha_T(V_S^{n-1} f)\alpha_W(V_S^{n-1} f) V_S^n f \quad \text{a.e.},$$
$$(V_Q V_T)^n V_S f = \alpha_T(V_S f)\alpha_W(V_S f) \cdots \alpha_T(V_S^n f)\alpha_W(V_S^n f) V_S^{n+1} f \quad \text{a.e.},$$

we have

$$(V_Q V_T)^n \varphi = \varphi \text{ a.e.}$$

where $\varphi = f^{-1} V_S f$. Since QT is isomorphic to T and T is totally ergodic, $\varphi = 1$ a.e. Thus we obtain $V_S f = f$ a.e. for each $f \in \mathbf{0}(T)$, and so S is an identity.

Proposition 4. *If a metric automorphism T is totally ergodic and has discrete spectrum, then its commuting order is two.*

Proof. For $S_3 \in C_3(T)$, choose $S_2 \in C_2(T)$ so that $V_{S_3} V_T V_{S_3}^{-1} = V_{S_2} V_T$. By Proposition 2, there exist metric automorphisms W and S such that W has each element of $\mathbf{0}(T)$ as a proper function and V_S maps $\mathbf{0}(T)$ onto itself and $S_2 = SW$. By Proposition 3, S is the identity and so

$$V_{S_3} V_T V_{S_3}^{-1} f = V_{S_2} V_T f = \alpha_T(f)\alpha_W(f) V_S f = \alpha_T(f)\alpha_W(f) f \text{ a.e.},$$
$$V_T f = \alpha_T(f) f \text{ a.e.}$$

for each $f \in \mathbf{0}(T)$. Thus we observe that every $f \in \mathbf{0}(T)$ is a proper function of $S_3 T S_3^{-1} T^{-1}$, and by Proposition 1 we can conclude that $S_3 T S_3^{-1} T^{-1} \in C_1(T)$, and so $S_3 \in C_2(T)$.

Proposition 5. *If a metric automorphism T is totally ergodic and has discrete spectrum, then $C_0(T)$, $C_1(T)$, and $C_2(T)$ are subgroups of G.*

Proof. From the definition, $C_0(T)$ is a subgroup of G. Suppose $S_1, S_1' \in C_1(T)$. Then, for each $f \in \mathbf{0}(T)$, $V_{S_1} f = \alpha_{S_1}(f) f$ a.e., and $V_{S_1'} f = \alpha_{S_1'}(f)$ a.e. Therefore $V_{S_1'} V_{S_1^{-1}} f = \alpha_{S_1'}(f)\alpha_{S_1}(f)^{-1} f$ a.e. By Proposition 1, we observe that $S_1' S_1^{-1} \in C_1(T)$. It remains to show that $C_2(T)$ is a subgroup of G. Let $S_2 \in C_2(T)$ [$S_2' \in C_2(T)$]. Then there exist metric automorphisms W and S [W' and S'] such that W [W'] has each element of $\mathbf{0}(T)$ as a proper function and V_S [$V_{S'}$] maps $\mathbf{0}(T)$ onto itself and $S_2 = SW$ [$S_2' = S'W'$]. Thus we have

$$V_{S_2' S_2^{-1}}(V_S f) = V_{S_2'}(\alpha_W(f)^{-1} f) = \alpha_W(f)^{-1}\alpha_{W'}(f) V_{S'} f \text{ a.e.}$$

for each $f \in \mathbf{0}(T)$. Consequently

$$V_{(S_2' S_2^{-1}) T (S_2' S_2^{-1})^{-1} T^{-1}}(V_{S'} f) = \alpha_T(V_{S'} f)^{-1} \alpha_T(V_S f) V_{S'} f \text{ a.e.}$$

Every $f \in \mathbf{0}(T)$ is a proper function of $(S_2' S_2^{-1}) T (S_2' S_2^{-1})^{-1} T^{-1}$. Thus, by Proposition 1, we have $(S_2' S_2^{-1}) T (S_2' S_2^{-1})^{-1} T^{-1} \in C_1(T)$, that is, $S_2' S_2^{-1} \in C_2(T)$.

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