178. *On the Minimality of the Polar Decomposition in Finite Factors*

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(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 12, 1968)

1. Ky Fan and A. J. Hoffman [2] established the following matrix inequalities: For every unitarily invariant norm of matrices,

(i) If A is an $n\times n$ matrix and $A=UH$ where U is unitary and H is positive-definite, then

$$\|A-U\|\leqq\|A-W\|\leqq\|A+U\|,$$

for every unitary matrix W [2; Theorem 1],

(ii) If A is an $n\times n$ matrix, then

$$\left\|A-\frac{A+A^*}{2}\right\|\leqq\|A-H\|,$$

for every hermitean matrix H [2; Theorem 2],

(iii) If H and K are hermitean $n\times n$ matrices, then

$$\|(H-i)(H+i)^{-1}-(K-i)(K+i)^{-1}\|\leqq 2\|H-K\|,$$

[2; Theorem 3].

In this note, we shall extend these inequalities of Fan and Hoffman for finite factors.

2. Throughout the note, let $\mathcal{A}$ be a finite factor with the (normalized) faithful normal trace φ such that $\varphi(1)=1$ (cf. [1]). For each $T\in\mathcal{A}$,

$$\|T\|_2^2=\varphi(T^*T)$$

defines a norm on $\mathcal{A}$, by which $\mathcal{A}$ becomes a prehilbert space. In a finite factor $\mathcal{A}$, if $T=V|T|$ is the polar decomposition of T, then the partially isometric operator V can be extended to a unitary $U\in\mathcal{A}$ such that $T=U|T|$.

3. We shall show that the unitary operator U appeared in the polar decomposition is one of the nearest unitary operators to the given T in $\mathcal{A}$, which will give an illustration of the polar decomposition in the finite factor $\mathcal{A}$:

Theorem 1. *Let T be any operator in $\mathcal{A}$ and $T=UH$ the polar decomposition of T, where U is a unitary, then for any unitary operator V in $\mathcal{A}$,*

$$\|T-U\|_2\leqq\|T-V\|_2\leqq\|T+U\|_2. \tag{1}$$

Proof. By the definition of the norm,

$$\|T-U\|_2^2=\|UH-U\|_2^2=\varphi(H^2-2H+1),$$

and for a unitary operator $W \in \mathcal{A}$ such that $W = U^{-1}V$,

$$\|T-V\|_2^2 = \|UH - V\|_2^2 = \varphi(H^2 - HW - W^*H + 1).$$

Hence we have

$$\|T-V\|_2^2 - \|T-U\|_2^2 = 2\varphi(H) - \varphi(HW + W^*H)$$
$$= 2[\varphi(H) - \mathrm{Re}\ \varphi(HW)].$$

Now, $\varphi(H)$ is positive and

$$\begin{aligned} \mathrm{Re}\ \varphi(HW) &\leqq |\varphi(HW)| \\ &= |\phi(H^{\frac{1}{2}}H^{\frac{1}{2}}W)| \\ &\leqq \varphi(H)^{\frac{1}{2}}\varphi(W^*H^{\frac{1}{2}}H^{\frac{1}{2}}W)^{\frac{1}{2}} \\ &= \varphi(H), \end{aligned} \tag{2}$$

by the Schwarz inequality. Therefore,

$$\|T-V\|_2^2 - \|T-U\|_2^2 \geqq 0,$$

that is, we have proved the first inequality.

For the second inequality, we need the symmetric argument:

$$\|T+U\|_2^2 - \|T-V\|_2^2 = 2[\varphi(H) + \mathrm{Re}\ \varphi(HW)]$$

and (2) imply

$$\|T+U\|_2 \geqq \|T-V\|_2$$

for all unitary $V \in \mathcal{A}$.

4. We shall prove a converse of Theorem 1:

Theorem 2. *For an operator T in $\mathcal{A}$, let U be a unitary operator in $\mathcal{A}$ such that*

$$\|T-U\|_2 \leqq \|T-V\|_2$$

for any unitary operator V in $\mathcal{A}$, then $T = U|T|$.

Proof. Let $T = W|T|$ be a polar decomposition of T by a unitary operator W in $\mathcal{A}$.

By the assumption, we have

$$\|T-U\|_2 \leqq \|T-W\|_2.$$

Hence, we have

$$\varphi[(T-U)^*(T-U)] \leqq \varphi[(T-W)^*(T-W)],$$

and so

$$\varphi(W^*T + T^*W - U^*T - T^*U) \leqq 0.$$

Since φ is a faithful trace on $\mathcal{A}$,

$$0 \geqq \varphi(|T| + |T| - U^*W|T| - |T|W^*U)$$
$$= \varphi[|T|^{\frac{1}{2}}(U-W)^*(U-W)|T|^{\frac{1}{2}}] \geqq 0$$

implies

$$U|T|^{\frac{1}{2}} = W|T|^{\frac{1}{2}}.$$

Therefore, we have

$$T = W|T| = W|T|^{\frac{1}{2}}|T|^{\frac{1}{2}} = U|T|^{\frac{1}{2}}|T|^{\frac{1}{2}} = U|T|.$$

5. Since the proof of [2; Theorem 2] is based only on the invariance of the norm under the conjugation, (ii) of Fan and Hoffman is extendable in our case:

Theorem 3. *If $T \in \mathcal{A}$, then*

$$\left\| T - \frac{T+T^*}{2} \right\|_2 \leqq \| T - H \|_2, \tag{3}$$

for any hermitean $H \in \mathcal{A}$.

We shall repeat the proof of Fan and Hoffman:

$$\begin{aligned}\left\| T - \frac{T+T^*}{2} \right\|_2 &= \left\| \frac{T-H}{2} + \frac{H-T^*}{2} \right\|_2 \\ &\leqq \frac{1}{2} \| T-H \|_2 + \frac{1}{2} \| T^* - H \|_2 \\ &= \| T-H \|_2 .\end{aligned}$$

A converse of Theorem 3 will be obtained in a forthcoming paper of T. Furuta and R. Nakamoto.

6. For (iii), we have also

$$\left\| \frac{H-i}{H+i} - \frac{K-i}{K+i} \right\|_2 \leqq 2 \| H-K \|_2, \tag{4}$$

for every pair of hermitean operators H *and* K *belonging to* $\mathcal{A}$. However, we do not give here a proof, since (4) is already established by Murray and von Neumann [3; Lemma 1.5.1].

References

[1] J. Dixmier: Les algebres d'operateurs dans l'espace Hilbertien. Gauthier-Villars, Paris (1957).

[2] Ky Fan and A. J. Hoffman: Some metric inequalities in the space of matrices. Proc. Amer. Math. Soc., **6**, 111–116 (1955).

[3] F. J. Murry and J. von Neumann: Rings of operators. IV. Ann. of Math., **44**, 716–808 (1943).