228. *On a Theorem on Commutative Decompositions*

By Kiyoshi ISÉKI

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J. R. Büchi [1] introduced a useful notion called *a pair of functions* (f, f'). Let E, E' be sets, and let $f: 2^E \to 2^{E'}$, $f': 2^{E'} \to 2^E$ be functions. Then (f, f') is a pair of functions, if $A' \cap f(A) = \phi$ implies $f'(A') \cap A = \phi$, where $A \subset E$, $A' \subset E'$. As shown by J. R. Büchi [1], an equivalence relation or a decomposition of E is defined by a pair of functions (f, f').

Let (f, f') be a pair of functions from 2^E to 2^E. If 1) $A \subset f(A)$, 2) $f(A) = f'(A)$, and 3) $f(f(A)) \subset f(A)$ for every $A \subset E$, then (f, f') or f is called an *equivalence relation.*

In my note [2], we discussed some classical results on mappings by the notion of pair of functions. In this Note, we shall consider Sik results on the equivalence relations [3].

Theorem. *Let f, g be two equivalence relations on a set E. The following propositions are equivalent.*

1) *The composition fg is an equivalence relation.*

2) *for any subsets A, B of E, $f(A) \cap g(B) = \phi$ implies $g(A) \cap f(B) = \phi$.*

3) *for any subsets A, B of E, $f(A) \cap g(B) \neq \phi$ implies $g(A) \cap f(B) \neq \phi$.*

4) *for any subset A of E, $fg(A) = gf(A)$.*

Proof. It is obvious that the conditions 2) and 3) are equivalent.

To prove 3)$\Rightarrow$4), let $x \in fg(A)$, then

$$x \cap f(g(A)) \neq \phi.$$

Hence $f(x) \cap g(A) \neq \phi$. From 3), we have $g(x) \cap f(A) \neq \phi$, which means $x \in gf(A)$. Therefore $fg(A) \subset gf(A)$. Similarly we have $gf(A) \subset fg(A)$.

To prove 4)$\Rightarrow$3), suppose that $f(A) \cap g(B) \neq \phi$, then $A \cap fg(B) \neq \phi$. By 4), we have $A \cap gf(B) \neq \phi$, and then $g(A) \cap f(B) \neq \phi$.

Therefore 3) and 4) are equivalent.

Next we shall prove 1)$\Rightarrow$2).

Let $f(A) \cap g(B) = \phi$, then we have

$$A \cap fg(B) = \phi.$$

Therefore $(fg)'(A) \cap B = \phi$. Since fg is the equivalence relation, $(fg)'$ fg. Hence $fg(A) \cap B = \phi$, and then $g(A) \cap f(B) = \phi$, which shows 3).

Finally we show 4)$\Rightarrow$1). We must verify the three conditions of an equivalence relation.

1) Since f, g are two equivalence relations, for any subsets A, B of E, then

$$A \subset g(A) \subset f(g(A)).$$

2) To prove $(fg)' = fg$, consider

$$(fg)'(A) \cap B = \phi,$$

then we have

$$A \cap fg(B) = \phi.$$

By the condition 4), we have

$$A \cap gf(B) = \phi.$$

Since f, g are the equivalence relations, we have $g(A) \cap f(B) = \phi$ and then $fg(A) \cap B = \phi$. Therefore $fg(A) \subset (fg)'(A)$.

Conversely, let $fg(A) \cap B = \phi$, then by the condition 4), we have $gf(A) \cap B = \phi$. This implies $(fg)'(A) \cap B = \phi$. Hence $(fg)'(A) \subset fg(A)$.

3) $(fg)((fg)(A)) \subset fg(A)$ follows from the following relation. By the condition 4), we have

$$fgfg(A) = ffgg(A) \subset fgg(A) = ggf(A) \subset gf(A) = fg(A).$$

Therefore we complete the proof of Theorem.

References

[1] J. R. Büchi: Die Boole'sche Partialordnung und die Paarung von Gefügen. Portugaliae Math., **7**, 119–180 (1948).

[2] K. Iséki: General theory of mappings. Proc. Japan Acad., **44**, 663–666 (1968).

[3] F. Silk: Über Charakterisierung kommutativer Zerlegungen. Publ. Fac. Sci. Univ. Masaryk., A, **354** (9), 1–6 (1953–1954).