227. *Pseudo Quasi Metric Spaces*

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Introduction. Kelly [3] is the first one who studied the theory of bitopological space. A motivation for the study of bitopological spaces is to generalize the pseudo quasi metric space (which we denote as $p-q$ metric). In this paper one observes the relation between $p-q$ metric spaces and the bitopological spaces which are generated by them. In chapter 2, one defines p-complete normal (i.e., pairwise complete normal) space and shows that $p-q$ metric space is p-complete normal. In the last chapter the $p-q$ metrisable problem is considered, and one of the Sion and Zelmer's result [4] is proved directly by a bitopological method. Throughout notations and definitions follow [2] and [3].

Definition. A $p-q$ metric on set X is a non-negative real valued function $p: X\times X\rightarrow R$ (reals) such that

(1) $p(x, x)=0$,

(2) $p(x, z)\leq p(x, y)+p(y, z)$ for all $x, y, z\in X$.

In addition, if p satisfies

(3) $p(x, y)=0$ only if $x=y$

then p is said to be a quasi metric. If p satisfies

(4) $p(x, y)=p(y, x)$

with (1) and (2) then p is a pseudo metric. Obviously, if (1), (2), (3), and (4) are satisfied then it is a metric in the usual sense.

Let p be a $p-q$ metric on X and let q be defined by $q(x, y)=p(y, x)$. Then q is a $p-q$ metric on X and q is said to be the conjugate $p-q$ metric of p. We denote the bitopological space X generated by $\{S_p(x, \varepsilon)=\{y\mid p(x, y)<\varepsilon\}\}$ and $\{S_q(x, \varepsilon)=\{y\mid q(x, y)<\varepsilon\}\}$ as (X, P, Q) (see [3]). Throughout this paper (X, L_1, L_2) denotes a bitopological space with topology L_1 and L_2.

(1–2) **Definition** (Kelly [3]). A bitopological space (X, L_1, L_2) is said to be p-normal (i.e., pairwise normal) if for any L_1-closed set A and L_2-closed set B with $A\cap B=\phi$, there exist an L_2-open U and an L_1-open set V such that $A\subset U$, $B\subset V$, and $U\cap V=\phi$.

Kelly [3] defined p-regular bitopological space in an analogous manner.

(1–3) **Definition.** Let (X, L_1, L_2) be a bitopological space,

(1) It is a $p-T_{1\frac{1}{2}}$ iff for $x, y \in X$, $x \neq y$ there exist $U \in L_1$ and $V \in L_2$ such that either $x \in U$, $y \in V$ or $x \in V$, $y \in U$ and $U \cap V = \phi$.

(2) It is $p-T_2$ iff for $x, y \in X$, $x \neq y$ there exist $U \in L_i$ and $V \in L_j$ such that $i \neq j$, $i=1,2$, $x \in U$, $y \in V$ and $U \cap V = \phi$.

The definition of $p-T_2$ was given by Weston [5]. It is obvious from the definition that $p-T_2$ implies $p-T_1$. Further if (X, L_1, L_2) is a $p-T_2$ space the (X, L_i) is a T_1-space and if (X, L_1, L_2) is a $p-T_{1\frac{1}{2}}$ space then (X, L_i) is a T_0-space for $i=1,2$.

(1–4) **Definition.** A $p-q$ metric p is called a $A-p-q$ metric (Albert $p-q$ metric [1]) if it satisfies the condition that $x \neq y$ implies either $p(x,y) \neq 0$ or $q(x,y) \neq 0$.

It is easy to prove the following

(1–5) **Theorem.** *If (X, P, Q) is generated by the $A-p-q$ metric p and its conjugate metric q, respectively, then it is $p-T_1$.*

Remark. Similarly, (X, P, Q) is $p-T_2$ iff it is quasi metric (see [3]).

The following is an example for $A-p-q$ metric.

(1–6) **Example.** Let X be the set of all reals and

$$p(x,y) = |x-y| \quad \text{if } x<y$$
$$= 0 \quad \text{if otherwise}$$

and $q(x,y)=p(y,x)$. Then (X, P, Q) is $p-T_{1\frac{1}{2}}$ but it is not $p-T_2$.

(1–7) **Theorem** (Kelly [3]). *A $p-q$ metric (X, P, Q) is p-regular and p-normal.*

2. In this chapter one defines p-complete normality and shows that a $p-q$ metric space (X, P, Q) is p-complete normal.

(2–1) **Definition.** In a bitopological space (X, L_1, L_2) a pair (A, B), $A, B \subset X$ is said to be (12)-*separated* iff $\bar{A} \cap B = A \cap \bar{\bar{B}} = \phi$, where $\bar{A}$ is the L_1-closure of A and $\bar{\bar{B}}$ is the L_2-closure of B.

Remark. If $L_1 \subset L_2$ then (12)-separated implies L_2-separated.

(2–2) **Definition.** A bitopological space (X, L_1, L_2) is said to be *p-completely normal* iff for every (12)-separated pair (A, B) there exist an L_2-open set $U \supset A$ and an L_1-open set $V \supset B$ such that $U \cap V = \phi$.

(2–3) **Theorem.** *A bitopological space (X, L_1, L_2) is p-completely normal iff every subset of X is p-normal.*

Proof. Suppose X is p-completely normal and $Y \subset X$. Let F_1 and F_2 be disjoint closed (relative to Y) in L_1 and L_2, respectively. Then $F_1 \cap \bar{\bar{F}}_2 = \bar{F}_1^{L_{Y_1}} \cap \bar{\bar{F}}_2 = Y \cap \bar{F}_1 \cap \bar{\bar{F}}_2 = \bar{F}_1^{L_{Y_1}} \cap \bar{F}_2^{L_{Y_2}} = F_1 \cap F_2 = \phi$. Where $\bar{F}^{L_{Y_i}}$ denotes the L_{Y_i}-closure of F. Similarly, we can show $\bar{F}_1 \cap F_2 = \phi$ which implies (F_1, F_2) is a (12)-separated pair of X. By p-complete normality there exist disjoint sets L_1-open G_1 and L_2-open G_2 containing F_2 and F_1, respectively. Then $Y \cap G_1$ and $Y \cap G_2$ are disjoint L_{Y_1}, L_{Y_2} open sets of Y which contain F_2 and F_1, so that Y is a p-normal space.

Conversely, let (A,B) be a (12)-separated pair, i.e.,

$$(\bar{A}\cap B)\cup(A\cap\bar{\bar{B}})=\phi.$$

Let $Y=(\bar{A}\cap\bar{\bar{B}})^c$. Then (Y, L_{Y_1}, L_{Y_2}) is a p-normal space by assumption since

$$Y\cap\bar{A}=(\bar{A}^c\cup\bar{\bar{B}}^c)\cap\bar{A}=\bar{\bar{B}}^c\cap\bar{A},\qquad Y\cap\bar{\bar{B}}=(\bar{A}^c\cup\bar{\bar{B}}^c)\cap\bar{\bar{B}}=\bar{A}^c\cap\bar{\bar{B}}$$

then $Y\cap\bar{A}$ and $Y\cap\bar{\bar{B}}$ are disjoint L_{Y_1} and L_{Y_2}-closed sets, respectively. Therefore, there exist $U\cap Y=U_Y\in L_{Y_1}$ and $V\cap Y=V_Y\in L_2$ such that $(Y\cap\bar{\bar{B}})\subset U_Y$ and $(Y\cap\bar{A})\subset V_Y$, where $U_Y\cap V_Y=\phi$. But $U\cap(\bar{A}\cap\bar{\bar{B}})^c$ $=U\cap(\bar{A}^c\cup\bar{\bar{B}}^c)$.

$$U_Y=U\cap Y=(U\cap\bar{\bar{B}}^c)\cup(U\cap\bar{A}^c)\text{ where } U\in L_1,$$

$$V_Y=V\cap Y=(V\cap\bar{\bar{B}}^c)\cup(V\cap\bar{A}^c)\text{ where } V\in L_2.$$

Since $(U\cap\bar{\bar{B}}^c)\cap\bar{\bar{B}}=\phi$,

$$U_Y\supset(Y\cap\bar{\bar{B}})\text{ implies }(U\cap\bar{A}^c)\supset(Y\cap\bar{\bar{B}}).$$

Similarly $V_Y\supset(Y\cap\bar{A})$ implies $(V\cap\bar{\bar{B}}^c)\supset(Y\cap\bar{A})$.

Now, $U'=U\cap\bar{A}^c\in L_1$ and $V'=V\cap\bar{\bar{B}}^c\in L_2$ and $U'\cap V'=\phi$. Consider

$$Y\cap\bar{\bar{B}}=(\bar{A}^c\cup\bar{\bar{B}}^c)\cap\bar{\bar{B}}=A^c\cap\bar{\bar{B}}.$$

But $\bar{A}\cap B=\phi$ so that $\bar{A}^c\supset B$. Therefore

$$\bar{A}^c\cap\bar{\bar{B}}\supset B\text{ and } U'=U\cap\bar{A}^c\supset(Y\cap\bar{\bar{B}})=\bar{A}\cap\bar{\bar{B}}\supset B.$$

Similarly,

$$V'\supset(\bar{\bar{B}}^c\cap\bar{A})\supset A.$$

(2–4) **Lemma.** *Every subspace of a $p-q$ metric space (X,P,Q) is a $p-q$ metric space.*

(2–5) **Theorem.** *A $p-q$ metric space (X,P,Q) is p-completely normal.*

Proof. By (1–7) a $p-q$ metric space is p-normal, and by the above lemma every subspace of a $p-q$ metric space is a $p-q$ metric space also, which implies that every subspace is p-normal. Apply (2–3) and the statement is proved.

3. In this chapter $p-q$ metrisable theorems are considered in the context of bitopological spaces and Sion and Zelmer's result [4] will be proved in a direct way. We start with a few lemmas which will be used in the sequel.

(3–1) **Definition.** In a bitopological space (X,L_1,L_2) a subset $C\subset X$ is said to be (12)-*disjoint* iff for each $x\in C$ and $y\in C^c$ there exist $U_x\in L_1$ and $V_y\in L_2$ such that $x\in U_x$, $y\in V_y$ and $U_x\cap V_y=\phi$. A set both (12)-disjoint, (21)-disjointed is called *p*-disjoint*.

Remark. In the example (1–6) every L_2-closed set is (12)-disjoint and every L_1-closed set is (21)-disjoint. If (X,L_1,L_2) is p-Hausdorff, then every subset of X is p*-disjoint.

(3–2) **Lemma.** *If a bitopological space (X,L_1,L_2) is L_1-regular and p-regular then an L_i-closed set is (i,j)-disjoint $(i\neq j,\ i,j=1,2)$.*

Proof. *Case* 1. Let C be an L_2-closed set and $x\notin C$. By p-

regularity there exist $U \in L_2$, $V \in L_1$ such that $C \subset V$, $x \in U$ and $V \cap U = \phi$.

For any $y \in C$, $y \notin V^c$ and $x \in V^c$ where V^c is L_1-closed. By the regularity of L_1 there exist $\alpha, \beta \in L_1$ such that

$$B \supset V^c, \quad y \in \alpha \text{ and } \alpha \cap \beta = \phi.$$

Then $y \in \alpha \subset \beta^c$ and $x \notin \beta^c$. Again by p-regularity there exist $W \in L_1$, $R \in L_2$ such that $y \in \beta^c \subset R$, $x \in W$ and $R \cap W = \phi$ which implies C is (21)-disjoint.

Case 2. If C is L_1-closed. The proof is similar to case 1.

Sion and Zelmer [4] proved the following theorem which we prove directly by a bitopological method.

(3–3) **Theorem.** *If (X, L_1) is regular, compact, $p-q$ metric topological space, then it is pseudo metric space.*

Proof. Let L_2 be the topology which is generated by $\{S_q(x, \varepsilon) = \{y : q(x, y) = p(y, x) < \varepsilon\}\}$, where L_1 is generated by the $p-q$ metric p. Then (X, L_1, L_2) is a $p-q$ space (or $(X, L_1, L_2) = (X, P, Q)$).

Let $U \in L_1$ then U^c is compact in L_1. By the lemma (3–2) U^c is (12)-disjoint and the compactness of U^c implies U^c is L_2-closed and $U \in L_2$. Therefore $L_1 \subset L_2$. $d(x, y) = \max\{p(x, y), q(x, y)\} = q(x, y)$ implies L_2 is a pseudo metric (by the symmetric property of q).

Similarly, we can show

(3–4) **Corollary.** *If (X, L_1) is a compact and quasi metric topological space, then it is a metric space.*

Proof. (X, P, Q) is $p-T_2$ iff it is quasi metrisable (see the remark following (1–5)) and every subset is (ij)-disjoint. Apply a similar method as (3–3) to complete the proof.

References

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