38. On Weak Convergence of Transformations in Topological Measure Spaces. II[1)]

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The author extended slightly in [4] a theorem of F. Papangelou [2, Theorem 2] as follows: Let X be a metrizable locally compact space and μ a σ-finite Radon measure on X. Then a sequence $\{T_n\}$ of invertible μ-measure-preserving transformations in X converges to an invertible μ-measure-preserving transformation T in X weakly if and only if every subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ has a subsequence $\{T_{k(l(n))}\}$ which converges to T almost everywhere.

In this paper we study weak convergence of a sequence $\{T_n\}$ of invertible μ-measure-preserving transformations in a metrizable space X with a tight measure μ. The theorems below generalize the first two theorems in [4].

Let (M, Ω, μ) be any measure space. The members of Ω are called measurable. If E is measurable then we will say that the measure space (E, Ω_E, μ_E), where $\Omega_E \equiv \{F \in \Omega \mid F \subset E\}$ and $\mu_E(F) \equiv \mu(F)$ for F which belongs to Ω_E, is a subspace of (M, Ω, μ).

Definition 1. A measure space (M, Ω, μ) is a σ-finite Lebesgue space if there exists a countable family $\Gamma = \{M_n\}$ of mutually disjoint measurable sets such that $0 < \mu(M_n) < \infty$, $\bigcup_n M_n = M$, and $(M_n, \Omega_{M_n}, \mu_{M_n})$ is a Lebesgue space in the sense of V. A. Rohlin [3] for each n.

Proposition 1. *If (M, Ω, μ) is a σ-finite Lebesgue space then there exist a locally compact, σ-compact, metrizable space H containing M and a Radon measure ν on H which satisfy the following properties:*

(i) *M is a ν-measurable subset of H and $\nu(H - M) = 0$.*

(ii) *(M, Ω, μ) is a subspace of $(H, \mathfrak{M}, \nu)$, where $\mathfrak{M}$ is the σ-field of subsets of H which are ν-meaurable.*

Moreover if $\Gamma_n = \{\Gamma_{nj}\}$ is a basis of the Lebesgue space $(M_n, \Omega_{M_n}, \mu_{M_n})$, where $\{M_n\}$ is a countable measurable decomposition of M such that $(M_n, \Omega_{M_n}, \mu_{M_n})$ is a Lebesgue space for each n, then the sets of the form $\bigcap_{j=1}^{N} A_{nj}$, where A_{nj} stands for one of the two sets Γ_{nj} and $M_n - \Gamma_{nj}$, can be taken as the topological open basis of the topological subspace

1) Continued from the paper No. 10 in Proc. Japan Acad., **45**, 39–44 (1969).

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Proof. Let $\Gamma_n=\{\Gamma_{nj}\}$ be a basis of the Lebesgue space M_n. Then there exists a complete measure space $(\tilde{M}_n, \tilde{\Omega}_n, \tilde{\mu}_n)$ with basis $\tilde{\Gamma}_n=\{\tilde{\Gamma}_{nj}\}$ having the following properties:

(iii) $M_n \subset \tilde{M}_n$, $M_n \in \tilde{\Omega}_n$, $\tilde{\mu}_n(M_n - M_n)=0$, $\tilde{\Gamma}_{nj}\cap M_n=\Gamma_{nj}$ for each j, and $(M_n, \Omega_{M_n}, \mu_{M_n})$ is a subspace of $(\tilde{M}_n, \tilde{\Omega}_n, \tilde{\mu}_n)$.

(iv) All sets of the form $\bigcap_j \tilde{A}_{nj}$, where $\tilde{A}_{nj}$ stands for one of the two sets $\tilde{\Gamma}_{nj}$ and $\tilde{M}_n - \tilde{\Gamma}_{nj}$ and j runs through all possible values, are nonvoid.

If we take the family of the sets of the form $\bigcap_{j=1}^{N} \tilde{A}_{nj}$ as the topological basis of the topology of $\tilde{M}_n$ then it is easy to see that $\tilde{M}_n$ is homeomorphic to the direct product topological group H_n of the countably many copies of the cyclic group $Z(2)$ of order 2 with discrete topology. Hence $\tilde{M}_n$ is a compact metrizable space with respect to this topology. Then it is not difficult to see that $\tilde{\mu}_n$ is just a Radon measure on $\tilde{M}_n$. In fact the σ-field generated by the sets of the form $\bigcap_{j=1}^{N} \tilde{A}_{nj}$ coincides with the σ-field $\tilde{\mathfrak{B}}$ generated by the open subsets of $\tilde{M}_n$ and $\tilde{\mu}_n$ on $\tilde{\Omega}_n$ is the completion of $\tilde{\mu}_n$ on $\tilde{\mathfrak{B}}$. Since $\tilde{M}_n$ is a compact metrizable space, it follows now that for any measurable subset E of $\tilde{M}_n$ and for any $\varepsilon>0$ there exist a compact set K and an open set G in $\tilde{M}_n$ such that $K\subset E\subset G$ and $\tilde{\mu}_n(G-K)<\varepsilon$ (refer to [1, Theorem 1.1]). This implies that $\tilde{\mu}_n$ is a Radon measure on $\tilde{M}_n$.

Put $H=\bigcup_n \tilde{M}_n$, where we can assume that all $\tilde{M}_n$ are mutually disjoint. We will define a subset 0 of H to be open in H if $\tilde{M}_n \cap 0$ is an open subset of $\tilde{M}_n$ for each n. Then obviously H is a locally compact, σ-compact, metrizable space. If we define a measure ν on the σ-field $\tilde{\Omega}\equiv\left\{\bigcup_n E_n \,|\, E_n\in\tilde{\Omega}_n\right\}$ of subsets of H as follows:

$$\nu\left(\bigcup_n E_n\right)=\sum_n \tilde{\mu}_n(E_n)$$

for $\bigcup_n E_n \in \tilde{\Omega}$, where E_n belongs to $\tilde{\Omega}_n$ for each n. Then it follows that ν is a Radon measure on H. In fact, for every $E\in\tilde{\Omega}$ we have

$$\begin{aligned}\nu(E)&=\sup\{\nu(K)\,|\,K\subset E \text{ and } K \text{ is compact}\}\\&=\inf\{\nu(G)\,|\,E\subset G \text{ and } G \text{ is open}\}.\end{aligned}$$

The proposition is now obvious.

Let (M, Ω, μ) be any measure space. We denote by $\mathfrak{G}$ the group of all invertible μ-measure-preserving transformations in M.

Definition 2. The sequence $\{T_n\}$ in $\mathfrak{G}$ converges to T in $\mathfrak{G}$ weakly if $\lim_{n\to\infty} \mu(T_nE+TE)=0$ for every measurable subset E of M with $\mu(E)<\infty$.

Proposition 2. *Let (M, Ω, μ) be a σ-finite Lebesgue space. If T, T_n $(n=1,2,3,\cdots)$ are in $\mathfrak{S}$ then* (a) *and* (b) *below are equivalent:*

(a) *$\{T_n\}$ converges to T weakly.*

(b) *Every subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ has a subsequence $\{T_{k(l(n))}\}$ which converges to T almost everywhere with respect to the topology given in the second half of Proposition* 1.

Proof. The proof is obvious by virtue of Proposition 1 and [2, Theorem 2].

Let X be any metrizable space and $\mathfrak{B}$ the σ-field generated by the open subsets of X. The members of $\mathfrak{B}$ are called the Borel subsets of X. A complete measure μ on a σ-field $\mathfrak{M}$ of subsets of X is a tight measure on X if $\mathfrak{B}\subset\mathfrak{M}$, μ on $\mathfrak{M}$ is the completion of μ on $\mathfrak{B}$, and

$$\mu(E)=\sup\{\mu(K)\,|\,K\subset E \text{ and } K \text{ is compact}\}$$

for every measurable set of finite measure. μ is locally finite if any point of X has an open neighborhood with finite measure.

Proposition 3. *Let X be a separable, complete metric space. If a measure μ on $\mathfrak{M}$ is the completion of a σ-finite measure on $\mathfrak{B}$ then μ is a tight measure on X.*

Proof. There exists a countable family $\{E_n\}$ of mutually disjoint Borel sets such that $0<\mu(E_n)<\infty$ for each n and $\bigcup_n E_n=X$. We define a measure λ_n on $\mathfrak{B}$ as follows: $\lambda_n(E)=\mu(E_n\cap E)$ $(E\in\mathfrak{B})$. Then λ_n is a finite measure on $\mathfrak{B}$, and so for any Borel set E and for any positive number ε there exist a compact set K and an open set G in X such that $K\subset E\subset G$ and $\lambda_n(G-K)<\varepsilon$ (refer to [1, Theorems 1.1 and 1.4]).

Let A be a measurable set of finite measure. We may assume without loss of generality that A is a Borel set. If $\varepsilon>0$ is arbitrarily fixed then there exists a positive integer N such that $\mu\left(\bigcup_{n=1}^N A_n\right)>\mu(A)-\varepsilon$, where $A_n\equiv A\cap E_n$ for each n. Hence

$$\sum_{n=1}^N \lambda_n(A_n)>\mu(A)-\varepsilon.$$

Thus there exist compact sets K_n $(n=1,2,\cdots,N)$ such that $K_n\subset A_n$ and $\lambda_n(A_n-K_n)<\varepsilon/N$ for each n $(n=1,2,\cdots,N)$. If we put $K=\bigcup_{n=1}^N K_n$ then we have

$$\mu(A-K)=\mu\left(A-\bigcup_{n=1}^N A_n\right)+\mu\left(\bigcup_{n=1}^N A_n-K\right)$$
$$<\varepsilon+\sum_{n=1}^N \lambda_n(A_n-K_n)<2\varepsilon.$$

This completes the proof.

Let T_1 and T_2 be mappings of X into itself. Then we define the mapping denoted by $T_1\times T_2$ of X into $X\times X$ as follows: $(T_1\times T_2)x=(T_1x, T_2x)$ $(x\in X)$.

Lemma. *Let X be a metrizable space and μ a σ-finite tight measure on X. If T_1 and T_2 are measure-preserving transformations of $(X, \mathfrak{M}, \mu)$ into itself then the inverse image $(T_1 \times T_2)^{-1}(B)$ of any Borel subset B of $X \times X$ is a measurable subset of X.*

Proof. There exists a countable family $\{X_n\}$ of mutually disjoint measurable sets of finite measures such that $\bigcup_n X_n = X$. Then by the tightness of μ there exists a countable family $\{K_j\}$ of mutually disjoint compact sets of finite measures such that $\mu\left(X - \bigcup_j K_j\right) = 0$. Thus an argument analogous to that in the proof of [4, Lemma 2] suffices.

Theorem 1. *Let X be a metrizable space and μ a σ-finite tight measure on X. If T, T_n $(n=1, 2, 3, \cdots)$ are in $\mathfrak{G}$ then (α) below implies (β). In addition, if μ is locally finite then (α) and (β) are equivalent:*

(α) *$\{T_n\}$ converges to T weakly.*

(β) *Every subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ has a subsequence $\{T_{k(l(n))}\}$ which converges to T almost everywhere.*

Proof. *(α) implies (β)*: There exists a countable family $\{K_n\}$ of mutually disjoint compact sets such that $0 < \mu(K_n) < \infty$, $\mu\left(X - \bigcup_n K_n\right) = 0$. It is well-known that $(K_n, \Omega_{K_n}, \mu_{K_n})$ is a Lebesgue space and a countable open basis $\Gamma_n = \{\Gamma_{nj}\}$ can be taken as a basis of $(K_n, \Omega_{K_n}, \mu_{K_n})$ (see [3]). Hence $\left(\bigcup_n K_n, \Omega_{\bigcup_n K_n}, \mu_{\bigcup_n K_n}\right)$ is a σ-finite Lebesgue space. We can now see T, T_n $(n=1, 2, 3, \cdots)$ as invertible $\mu_{\bigcup_n K_n}$-measure-preserving transformations in $\bigcup_n K_n$. By virtue of Proposition 2, we have that if $\{T_n\}$ converges to T weakly there exists a subsequence $\{T_{k(n)}\}$ of $\{T_n\}$ which converges to T almost everywhere with respect to the topology induced by the topological basis of the sets of the form $\bigcap_{j=1}^{N} A_{nj}$, where A_{nj} stands for one of the two sets Γ_{nj} and $K_n - \Gamma_{nj}$. This topology is obviously finer than the topology of the topological subspace $\bigcup_n K_n$ of the metrizable space X. This implies that $\{T_{k(n)}\}$ converges to T almost everywhere with respect to the topology of X.

If μ is locally finite then (β) implies (α): By virtue of Lemma, the proof runs on the same line as that of the corresponding part of [2, Theorem 2], and so we omit the proof here.

Theorem 2. *Let X be a metrizable space and μ a σ-finite, locally finite tight measure on X. Let $\mathfrak{G}$ be the group of all automorphisms of the measure space $(X, \mathfrak{M}, \mu)$. Then weak topology on $\mathfrak{G}$ is the finest topology $\mathfrak{T}$ such that if a sequence $\{T_n\}$ in $\mathfrak{G}$ converges to T in $\mathfrak{G}$ almost everywhere then $\mathfrak{T}-\lim T_n = T$.*

Proof. By virtue of Theorem 1, the proof is analogous to that of [2, Theorem 3].

Remark 1. In the first half of Theorem 1 the σ-finiteness of μ is not omitted. To see this, let X be the unit interval $[0, 1]$ with discrete topology. We define μ on the family $\mathfrak{M}$ of all subsets of X as follows

$$\mu(E)=\begin{cases} 0 & \text{if } E \text{ is a finite set} \\ \infty & \text{if } E \text{ is an infinite set} \end{cases} \qquad (E \in \mathfrak{M}).$$

Then μ is a tight measure on X, but not σ-finite. Let T be an arbitrary one to one onto transformation in X such that $Tx \neq x$ for all x. Then we put $T_n = T$ $(n=1, 2, 3, \cdots)$. It is obvious that $\{T_n\}$ converges to any invertible μ-measure-preserving transformation T in X weakly. However for any x in X $\lim T_n x = Tx \neq x = Ix$, where I is the identity transformation in X.

Remark 2. In the second half of Theorem 1 the locally finiteness of μ is not omitted. To see this, let $X=\{0, 1, 1/2, 1/3, \cdots\}$. X is a compact metric space as a subspace of the real line. We define μ on the σ-field $\mathfrak{B}$ of all Borel subsets of X as follows:

$$\mu(E)=\begin{cases} n & \text{if } E \text{ contains } n \text{ elements} \\ \infty & \text{if } E \text{ is an infinite set} \end{cases} \qquad (E \in \mathfrak{B}).$$

Then μ is a σ-finite tight measure on X, but not locally finite. In fact, any open set containing 0 is of infinite measure. Let T_n be an invertible μ-measure-preserving transformation in X defined as follows:

$$T_n x=\begin{cases} 1/n & \text{if } x=0 \\ 0 & \text{if } x=1/n \\ x & \text{if } x \neq 0,\ 1/n \end{cases} \qquad (x \in X).$$

Then $\lim_{n\to\infty} T_n x = x$ for every x in X. But if we put $E=\{0\}$ then

$$\mu(E) < \infty, \qquad \lim_{n\to\infty} \mu(T_n E + IE) = 2.$$

This implies that $\{T_n\}$ does not converge to the identity transformation I in X weakly.

Remark 3. Let (X, Ω, μ) and (Y, Σ, ν) be two σ-finite Lebesgue spaces. If U is an isomorphism of the measure ring (Ω, μ) associated with (X, Ω, μ) onto the measure ring (Σ, ν) associated with (Y, Σ, ν) then U is generated by the unique invertible measure-preserving transformation $T(U)$ of (X, Ω, μ) onto (Y, Σ, ν) mod zero. This is an easy application of the theory of V. A. Rohlin [3].

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References

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