

131. Structure Theorems for Some Classes of Operators

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1. We consider bounded linear operators on a Hilbert space H . Denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$, $\sigma_c(T)$ the spectrum, the point spectrum, the residual spectrum and the continuous spectrum respectively, by $W(T) = \{(Tx, x) : \|x\| = 1\}$ the numerical range. It is known [3] that $W(T)$ is convex and $\text{conv } \sigma(T) \subseteq \text{cl } W(T)$ ($\text{conv} = \text{convex hull}$, $\text{cl} = \text{closure}$). An operator T is said to be hyponormal if $T^*T - TT^* \geq 0$, or equivalently if $\|T^*x\| \leq \|Tx\|$ for every $x \in H$. As in [1] an operator is said to be restriction-convexoid (reduction-convexoid) if the restriction of T to every invariant (invariant under T and T^*) subspace is convexoid, where convexoid means that $\text{conv } \sigma(T) = \text{cl } W(T)$.

In this Note we give some theorems on structure of hyponormal and restriction-convexoid operators whose spectrum lies on a convex curve.

2. Our main result in this section is

Theorem 1. *If T is a hyponormal operator and has the following properties*

1° $T^p = ST^*pS^{-1} + C$ for some S for which $0 \notin \text{cl } W(S)$ and $C = \text{compact operator}$

2° if $\mu, \lambda \in \sigma(T)$, $1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots + \left(\frac{\lambda}{\mu}\right)^{p-1} \neq 0$

then T is a normal operator.

For the proof we need the following

Lemma 1. *If T is a hyponormal operator which is the sum of a self-adjoint operator A and a compact operator C , then T is a normal operator.*

Proof. We denote by $\sigma_r^*(T)$ the set of complex numbers λ such that $T - \lambda I$ has a continuous inverse and that the range of $T - \lambda I$ is not dense in H and $\sigma_c^*(T)$ is the set of complex numbers λ which does not belong to $\sigma_p(T)$ and for which there exists a sequence $\{x_n\}$ of unit vectors in H such that $\|Tx_n - \lambda x_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Since T is hyponormal it is known that T can be expressed uniquely as a direct sum $T = T_1 \oplus T_2$ defined on a product space $H = H_1 \oplus H_2$ where H_1 is spanned by all the proper vectors of T such that: (a) T_1 is normal and $\sigma(T_1) = \text{cl } \sigma_p(T)$, (b) T_2 is hyponormal and $\sigma_p(T_2) = \emptyset$, (c) T is normal if and only if T_2 is normal.

From the fact that $T=A+C$ we conclude by Lemma 2 [10] that $\sigma_c^*(T_2) \subseteq \sigma(A)$ and therefore $\sigma_c^*(T_2)$ is real. Since $\sigma_p(T_2) \cup \sigma_r^*(T_2) \cup \sigma_c^*(T_2) = \sigma(T_2)$ and $\sigma_r^*(T_2)$ is open and $\sigma(T_2)$ is closed, we have that $\partial_r^*(T_2) \subseteq \sigma_p(T_2) \cup \sigma_c^*(T_2) = \sigma_c^*(T_2)$ (∂^* =boundary of $\sigma_r^*(T_2)$). Therefore T_2 is self-adjoint since T_2 is hyponormal with real spectrum and the lemma is proved.

Proof of Theorem 1. If $I(H)$ is the ideal of compact operators and $\omega(T)$ the Weyl spectrum (this means $\omega(T) = \bigcap_{C \in I(H)} \sigma(T+C)$) we obtain by the same reason as in [4] that $\omega(T)$ is real. By a result of Coburn [2] we conclude that $\sigma(T) = \omega(T) \cup \sigma_0(T)$ where $\sigma_0(T)$ contains only isolated eigenvalues of finite multiplicity. Let T_1 the restriction of T to the space H_1 generated by eigenvectors corresponding to eigenvalues $\lambda \in \sigma_0(T)$. If we denote $H_2 = H_1^\perp$, we obtain

$$H = H_1 \oplus H_2$$

and if $C = T_1 \oplus 0$ and $A = 0 \oplus T_2$ we obtain $T = A + C$ where A is self-adjoint and C is compact (with finite range) and by Lemma 1 it follows that T is a normal operator.

Corollary. *If T is a hyponormal operator with compact imaginary part, then T is normal.*

It is easy to see that for every operator we have

$$T = T^* + 2i \operatorname{Im} T$$

and by Theorem 1 for $p=1$ the corollary follows.

Remark. It is easy to see that the corollary is equivalent to the Lemma 1. Indeed, given a hyponormal operator $T = A + C$ with $A^* = A$ and C compact, one has $T^* - T = C^* - C$ compact, thus T has compact imaginary part, and therefore T is normal by the Putnam-Yoshino result [7], [10].

3. Theorem. *If a reduction-convexoid operator T whose spectrum lies on a convex curve is the sum of a compact operator C and a generalized nilpotent operator Q then T is normal.*

Proof. Since T is convexoid and $\sigma(T)$ lies on a convex curve T can be expressed as a direct sum $T_1 \oplus T_2$ defined on a product space $H_1 \oplus H_2$, where H_1 is spanned by all the eigenvectors of T , such that T_1 is normal with $\sigma(T_1) = \operatorname{cl} \sigma_p(T)$. By Weyl's Theorem [3, Problem 143] we conclude that $\sigma(T) \subseteq \sigma(Q) = \{0\}$ except the eigenvalues which implies $\sigma(T_2) \subseteq \{0\}$ since $T = C + Q$ and H_2 is an invariant subspace of T and thus $\sigma(T_2) = \sigma_c(T_2) \subseteq \sigma(Q)$. By Lemma 6 [6], H_1 reduces T and thus T_2 is convexoid operator with a single point in the spectrum. Since this point is zero we conclude that $T_2 = 0$ which implies $H_2 = 0$ and T is a normal operator.

We recall that an operator T , $\|T\| \leq 1$ and $\sigma(T) \subseteq \{z : |z| = 1\}$ is called unimodular contraction.

Corollary 1. *If T is a convexoid unimodular contraction and $T=C+Q$ then T is a unitary operator.*

Corollary 2. *If a reduction-convexoid operator T with compact imaginary part has the spectrum on a convex curve, then T is a normal operator.*

Proof. By Weyl's Theorem $\sigma(T)$ is real except the eigenvalues and we conclude as above that $\sigma(T_2)$ is real. Therefore T_2 is self-adjoint and T is a normal operator.

Theorem 3. *If T is spectral (in the sense of Dunford) and restriction-convexoid with compact imaginary part, then there exists a direct decomposition of H , $H=H_\infty+H_1+H_2+\dots$ such that*

- a) $H_i, i=1, 2, \dots$ is invariant under T
- b) $T|_{H_\infty}$ is scalar ($T|_{H_\infty}$ is the restriction of T to H_∞)
- c) $T|_{H_i}=\mu_i I, i=1, 2, \dots, \mu_i$ complex numbers.

Proof. Since T is almost normal [8], it follows that there exists a direct decomposition of H with properties a) and b) and

$$T|_{H_i}=\mu_i I+Q_i,$$

Q_i are compact nilpotent operators.

But $T|_{H_i}$ is convexoid at therefore $T|_{H_i}-\mu_i I$ is also convexoid with a single point in the spectrum. Then

$$T|_{H_i}-\mu_i I=0$$

Theorem 4. *If a restriction-convexoid operator whose spectrum lies on a convex curve is polynomially compact, then T is a normal operator.*

Proof. By Theorem 2 [6] we have that $T=T_1\oplus T_2$ as above with the same properties. Since H_2 is invariant under T and T is polynomially compact then $\sigma_c(T|_{H_2})\subseteq\{\lambda:p(\lambda)=0\}$ where $p(\cdot)$ is a polynomial with $p(T)=\text{compact}$. Therefore $\sigma(T_2)$ is a finite set and thus $H_2=\{0\}$. Indeed, in the contrary case, since T is restriction-convexoid we have that $\sigma(T_2)=\sigma_p(T_2)$ which is a contradiction.

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