129. On the Category of $L^1(G)\cap L^p(G)$ in $A^q(G)$*)

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1. Introduction and the main results.

Let G and $\hat{G}$ be two locally compact abelian groups in Pontrjagin duality. The integration with respect to a suitably normalized Haar measure on G is indicated by the expressions such as

$$\int_G f(x)\,dx \tag{1}$$

Let $C_c(G)$ denote the space of all continuous complex-valued functions on G each of which vanishes outside of some compact set, and $C_0(G)$ the set of continuous functions each of which vanishes at infinity. We shall denote $A^p(G)$ $(1\leq p<\infty)$ the space of functions f in $L^1(G)$ whose Fourier transforms $\hat{f}$ belong to $L^p(\hat{G})$ $(p\geq 1)$ and with the norm defined by

$$\|f\|^p=\|f\|_1+\|\hat{f}\|_p \tag{2}$$

where $\|f\|_1=\int_G |f(x)|\,dx$ and $\|\hat{f}\|_p=\left(\int_{\hat{G}} |\hat{f}(\hat{x})|^p\,d\hat{x}\right)^{1/p}$, $d\hat{x}$ denotes the integration with respect to Haar measure on $\hat{G}$. Clearly, $A^p(G)$ is a dense ideal in $L^1(G)$ and is a Banach algebra under convolution with the norm $\|\cdot\|^p$ (see Larsen, Liu and Wang [6]).

We denote T_1 and T_2 the Fourier transforms on $L^1(G)$ and $L^2(G)$ respectively. That is

$$T_1 f(\hat{x})=\int_G (-x,\hat{x})f(x)\,dx \tag{3}$$

and

$$\|T_1 f\|_\infty\leq\|f\|_1 \qquad \|T_2 f\|_2=\|f\|_2. \tag{4}$$

If $f\in C_c(G)$, the Fourier transform T is defined by the usual expression

$$Tf(\hat{x})=\int_G (-x,\hat{x})f(x)\,dx, \tag{5}$$

and $T_1 f=T_2 f=Tf$ for every $f\in C_c(G)$. Throughout this present note, we suppose essentially that $1<p<2$ and $1/p+1/q=1$. A. Weil [9; pp. 116–117] has shown, by using the convexity theorem of M. Riesz

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that the mapping T of (5) has the property that $Tf \in L^q(\hat{G})$ and

(6) $$\|Tf\|^q \leq \|f\|_p \quad \text{for } 1<p<2.$$

Thus T can be extended to a bounded linear transform T_p with domain $L^p(G)$ and range contained in $L^q(\hat{G})$ such that $T_p(L^p(G))$ is dense in $L^q(\hat{G})$ and

(7) $$\|T_p f\|_q \leq \|f\|_p$$

for any $f \in L^p(G)$ $(1<p<2)$ (cf. E. Hewitt [2]). Furthermore one sees that T_p is a one-to-one mapping. Indeed, for $f \in L^p(G)$, $\varphi \in L^p(\hat{G})$, one defines the bilinear form by

$$B(f, \varphi)=\int_G f\overline{T'_p(\varphi)}\, dx=\int_{\hat{G}} T_p(f)\dot{\varphi}\, d\hat{x}$$

where $T_p: L^p(G)\to L^q(\hat{G})$ and $T'_p: L^p(\hat{G})\to L^q(G)$ are well defined bounded linear mappings since $C_c(G)$ and $C_c(\hat{G})$ are dense in $L^p(G)$ and $L^p(\hat{G})$ (cf. Weil 9 pp. 116–117), we see that if $T_p f=0$, then $B(f, \varphi)=0$ for any φ and hence $f=0$. This shows the one to one property of T_p.

For any locally compact abelian group G, the set $\widehat{L^1(G)}$ of the Fourier transforms of the group algebra $L^1(G)$ is dense in $C_o(\hat{G})$. Furthermore $\widehat{L^1(G)}$ is either a dense set of the first category in $C_o(\hat{G})$ or all of the space $C_o(\hat{G})$ according as G is infinite or finite (see I. E. Segal [8]). In [8], Segal suggested a question that if G is a locally compact abelian group and $1<p<2$, then the Fourier transform maps $L^p(G)$ into a dense subset of $L^q(\hat{G})$ where $1/p+1/q=1$, which is a set of first category only if G is infinite. The affirmative answer to this question was given by E. Hewitt [2].

Recently, Larsen, Liu and Wang [6] investigated the algebra $A^p(G)=\{f \in L^1(G)\,;\ \hat{f}(\hat{x}) \in L^p(\hat{G})\}$. They have shown that $L^1(G)\cap L^2(G)=A^2(G)$ [6; Theorem 8] and stated a plausible conjecture that

(8) "$L^1(G)\cap L^p(G)=A^q(G)$ $(1<p<2,\ 1/p+1/q=1)$ is false"

Our purpose is to prove this conjecture. In fact we have the following further result.

Theorem 1. *Let G be a non-discrete locally compact abelian group and $1<p<2$, $1/p+1/q=1$. Then the set $L^1(G)\cap L^p(G)$ is a dense set of the first category in $A^q(G)$ with respect to the A^q-topology (defined in* (2)*) and the set of functions in $A^q(G)$ which are not in $L^1(G)\cap L^p(G)$ is a dense set of the second category.*

2. Some lemmas.

The proof of Theorem 1 is based on the construction in Hewitt [2]. We need some lemmas for the proof. Now we start from the following.

Lemma 2. *Let G be any locally compact abelian group and $1<p<2$, $1/p+1/q=1$. Then the set $L^1(G)\cap L^p(G)$ is a dense set in $A^q(G)$*

with respect to the $A^q(G)$*-topology (defined in* (2)).

Proof. It sufficies to show that for any $\varepsilon>0$ and $f\in A^q(G)$, there exists a function $h\in L^1(G)\cap L^p(G)$ such that

$$\|f-h\|^q<\varepsilon. \tag{9}$$

Let $f\in A^q(G)$. Then there is a sequence $\{f_n\}_{n=1}^\infty$ in $L^1(G)\cap L^p(G)$ such that $f_n\to f$ in L^1-topology as $n\to\infty$. Suppose that $\{e_\alpha\}$ is an approximate identity in $A^q(G)$ (see Lai [5; Theorem 1]). Then for each e_α,

$$f_n*e_\alpha\to f*e_\alpha \quad \text{in } L^1\text{-topology}$$

when $n\to\infty$. The same argument can be carried over as the proof of Theorem 2 in Lai [5], thus there exist indices n_0 and α_0 such that

$$\|f_{n_0}*e_{\alpha_0}-f\|^q<\varepsilon.$$

Since $e_{\alpha_0}\in A^q(G)\subset L^1(G)$ and $f_{n_0}\in L^1(G)\cap L^p(G)\subset L^p(G)$,

$$e_{\alpha_0}*f_{n_0}\in L^1(G)\cap L^p(G)$$

(see Hewitt [3] Corollary 3.3 or Hewitt and Ross [4] p. 298). Therefore $L^1(G)\cap L^p(G)$ is dense in $A^q(G)$. Q.E.D.

We need the following lemma which is analogous to S. Banach [1; Theorem 2 pp. 197–199] in the case of $L^q(0,1)$ on the real line.

Lemma 3. *Let G be a locally compact abelian group and $\{g_n\}_{n=1}^\infty$ be any sequence in $L^p(G)$ which converges weakly to zero, then there exists a subsequence $\{g_{nk}\}_{k=1}^\infty$ of $\{g_n\}_{n=1}^\infty$ such that*

$$\left\|\sum_{k=1}^m g_{nk}\right\|_p=\begin{cases}0(m^{1/2}) & \text{for } 2\le p\\ 0(m^{1/p}) & \text{for } 1<p\le 2.\end{cases}$$

Proof. We can prove by the same argument, mutatis mutandis, as that for Theorem 2 of [1; pp. 197–199] (cf. also [2]).

3. Proof of the main theorem.

In the following proof, we need only for $p>2$ in Lemma 3.

Proof of Theorem 1. Define a norm on $L^1(G)\cap L^p(G)$ by

$$\|f\|=\|f\|_1+\|f\|_p \tag{10}$$

for any $f\in L^1(G)\cap L^p(G)$. Then $E=\{L^1(G)\cap L^p(G);\ \|\cdot\|\}$ is a Banach space. Let Φ be an identity mapping of E into $A^q(G)$. Thus for $f\in E$

$$\|\Phi f\|^q=\|f\|^q=\|f\|_1+\|\hat{f}\|_q\le\|f\|_1+\|f\|_p=\|f\|$$

proves that Φ is a bounded linear mapping of E one to one into $A^q(G)$. We want to prove that $L^1(G)\cap L^p(G)\neq A^q(G)$. We proceed by contradiction, supposing that $L^1(G)\cap L^p(G)=A^q(G)$. Then the transformation is a bicontinuous mapping of E onto $A^q(G)$, there exists a positive constant $C(\ge 1)$ such that

$$\|f\|=\|\Phi^{-1}(\Phi f)\|\le C\|\Phi f\|^q$$

for all $f\in E$. That is

$$\begin{aligned}\|f\|_1+\|f\|_p&\le C(\|\Phi f\|_1+\|\widehat{\Phi f}\|_q)\\&=C\|f\|_1+C\|\hat{f}\|_q\end{aligned}$$

or

$$\|f\|_p\le C\|\hat{f}\|_q+(C-1)\|f\|_1. \tag{11}$$

the following construction is based on Hewitt [2] Lemma A.

If G is non discrete group, then the Haar measure μ of every open set U containing the identity is positive but it can be made arbitrarily small for appropriately chosen U. It is then apparent that there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of pairwise disjoints measurable sets in G such that $\mu(A_n)>0$ $(n=1, 2, \cdots)$ and $\lim_{n\to\infty} \mu(A_n)=0$. Write $\mu(A_n)=\alpha_n$ and define

(12) $$f_n(x)=\begin{cases}\alpha_n^{-1/p} & x \in A_n \\ 0 & x \notin A_n,\end{cases}$$

then it is easy to see that $f_n \in L^1(G)\cap L^p(G)$. This sequence $\{f_n\}_{n=1}^{\infty}$ converges weakly to zero in $L^p(G)$. To show this fact, we consider an arbitrary function $\varphi \in C_c(G)$, then we have

$$\left|\int_G f_n(x)\varphi(x)\,dx\right| \leq \sup_{x\in G}|\varphi(x)|\,\alpha_n^{1-\frac{1}{p}}$$

and thus $\lim_{n\to\infty}\int_G f_n(x)\varphi(x)\,dx=0$. Since $C_c(G)$ is dense in $L^q(G)$, f_n converges weakly to zero.

As $\alpha_n\to 0$ for $n\to\infty$, we then can choose a subsequence $\{A_{nk}\}_{k=1}^{\infty}$ of $\{A_n\}_{n=1}^{\infty}$ such that

(13) $$\alpha_{nk}<\frac{1}{2^{k/1-\frac{1}{p}}}.$$

It follows that the subsequence $\{f_{nk}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ converges weakly to zero in $L^p(G)$ and

(14) $$\|f_{nk_1}+f_{nk_2}+\cdots+f_{nk_m}\|_p=m^{1/p}$$

for all subsets $\{f_{nk_1}, f_{nk_2}, \cdots, f_{nk_m}\}$ of $\{f_{nk}\}_{k=1}^{\infty}$ $(m=1, 2, \cdots)$. Hence the sequence $\{T_p f_{nk}\}_{k=1}^{\infty}$ i.e. $\{\hat{f}_{nk}\}_{k=1}^{\infty}$ converges weakly to zero in $L^q(\hat{G})$. By Lemma 3, there exists a subsequence $\{\hat{f}_{nk_i}\}_{i=1}^{\infty}$ of $\{\hat{f}_{nk}\}_{k=1}^{\infty}$ and a constant A such that

(15) $$\|\hat{f}_{nk_1}+\hat{f}_{nk_2}+\cdots+\hat{f}_{nk_m}\|_q\leq Am^{1/2} \qquad (q>2).$$

Therefore, by (11),

$$\left\|\sum_{i=1}^{m} f_{nk_i}\right\|_p \leq C\left\|\sum_{i=1}^{m}\hat{f}_{nk_i}\right\|_q+(C-1)\left\|\sum_{i=1}^{m} f_{nk_i}\right\|_1.$$

It follows from (13)–(15) that

$$m^{1/p}\leq ACm^{1/2}+(C-1)\sum_{i=1}^{m}\alpha_{nk_i}^{1-\frac{1}{p}}$$
$$\leq ACm^{1/2}+(C-1)\sum_{i=1}^{m}\frac{1}{2^{k_i}}$$

or

$$m^{1/p-1/2}\leq AC+(C-1)\left(\sum_{i=1}^{m}\frac{1}{2^{k_i}}\right)m^{-1/2}.$$

This inequality holds only for $1/p-1/2\leq 0$, that is $p\geq 2$ and so it is a contradiction. This proves that $L^1(G)\cap L^p(G)\neq A^q(G)$. Therefore

$\Phi E \neq A^q(G)$. By open mapping theorem (cf. Kelley, $\cdots$, [7] p. 99), E is a set of the first category in $A^q(G)$. It combines with Lemma 2 that $L^1(G)\cap L^p(G)$ is a dense set of the first category in $A^q(G)$; since $A^q(G)$ is complete, the set of functions in $A^q(G)$ which are not in $L^1(G)\cap L^p(G)$ must be of the second category and accordingly dense. Q.E.D.

In Theorem 1 we assume that G is non discrete group, however if G is discrete topological group, then we disprove the conjecture (8). Hence we establish the following

Remark. If G is a discrete topological abelian group, then $A^q(G) = l^1(G)\cap l^p(G)$ for any $p, q\geq 1$.

Proof. As G is discrete, $\hat{G}$ is compact. It follows from that $l^1(G)=A^q(G)$ for any $q\geq 1$. And $l^1(G)\subset l^p(G)$ for any $p\geq 1$, we then have $l^1(G)\cap l^p(G)=l^1(G)=A^q(G)$. Q.E.D.

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