126. *Some Theorems on Certain Contraction Operators*

By Takayuki FURUTA*) and Ritsuo NAKAMOTO**)

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1969)

1. Let H be a complex Hilbert space. An operator T means a bounded linear operator on H. In this paper we shall prove some theorems on certain contraction operators and related results in consequence of these theorems.

We should like to express here our cordial thanks to Professor Masahiro Nakamura for his kind advice and for liberal use of his time in the preparation of this paper.

2. In this section, at first we shall begin to define the classes of operators as follows.

Definition 1. *An operator T is said to be normaloid in the sense that T satisfies*

(1) $$\|T^n\| = \|T\|^n \qquad n=1, 2, \cdots$$

equivalently, the spectral radius $r(T)$ is equal to $\|T\|$ ([3]).

Definition 2. *An operator T is said to be paranormal in the sense that T satisfies*

(2) $$\|T^2x\| \geqq \|Tx\|^2 \qquad \text{for every unit vector } x \text{ in } H.$$

In [4] this operator is named an operator of class (N).
It is known that this class of paranormal operators properly includes that of hyponormal operators and is properly included in the class of normaloids [1]-[3].

We shall discuss the following theorem and its consequence.

Theorem 1. *An idempotent normaloid operator T is a projection.*
To prove Theorem 1, we need the following already known theorem ([8]).

Theorem 2. *If T is an idempotent and contraction operator ($\|T\| \leqq 1$), then T is a projection.*
The following proof of Theorem 2, based on a method of Mlak [5], which is originally due to von Neumann, is simpler than that appeared in the literature.

Proof of Theorem 2.

$$\begin{aligned}\|Tx - T^*Tx\|^2 &= \|Tx\|^2 - (Tx, T^*Tx) - (T^*Tx, Tx) + \|T^*Tx\|^2\\ &= \|Tx\|^2 - (T^2x, Tx) - (Tx, T^2x) + \|T^*Tx\|^2\\ &= \|Tx\|^2 - \|Tx\|^2 - \|Tx\|^2 + \|T^*Tx\|^2\\ &= \|T^*Tx\|^2 - \|Tx\|^2 \leqq 0\end{aligned}$$

*) Faculty of Engineering, Ibaraki University, Hitachi.
**) Tennoji Senior High School, Osaka.

by $\|T^*\| \leqq 1$. *Hence* $T = T^*T$, *that is,* T *is self adjoint. Therefore* T *is a projection.*

Proof of Theorem 1. By means of Theorem 2, it is sufficient to show that $\|T\| \leqq 1$. If T is an idempotent normaloid, then we have

$$\|T\| = \|T^2\| = \|T\|^2$$

whence we can conclude $\|T\| \leqq 1$.

Every paranormal operator is a normaloid, so that we have the following corollary by Theorem 1.

Corollary 1. *An idempotent paranormal operator is a projection.*

3. In this section we shall give here a remark that Theorem 2 can be sharpened, namely we can weaken the idempotency of operator in Theorem 2.

Theorem 3. *If* T *is a contraction and satisfies*

$$T^k = T \tag{1}$$

for some positive integer $k \geqq 2$, *then* T^{k-1} *is a projection.*

Proof. $T^{2(k-1)} = T^{k-2}T^k = T^{k-2}T = T^{k-1}$, so T^{k-1} is an idempotent operator. Hence by Theorem 2, T^{k-1} is a projection.

Corollary 2. *If* T *is a normaloid and satisfies* (1), *then* T^{k-1} *is a projection.*

Proof. If T is a normaloid, then $\|T^k\| = \|T\|^k = \|T\|$, whence we have $\|T\| \leqq 1$. By means of Theorem 3, so the proof is complete.

Since Stampfli [6] established that

T is normal if T is hyponormal and T^k is normal for some k, so that Corollary 2 implies

Corollary 3. *If* T *is hyponormal and satisfies* (1), *then* T *is normal.*

Motivated by the above Corollary 3, the following theorem is naturally raised.

Theorem 4. *If* T *is a contraction and satisfies* (1), *then* T *is normal and partially isometric.*

Proof. Normality of T.

It is sufficient to show that $\|Tx\| = \|T^*x\|$. T is a contraction and T^{k-1} is a projection by Theorem 3, so that we have

$$\|Tx\| = T^k x\| = \|TT^{k-1}x\| = \|TT^{*k-1}x\| \leqq \|T\| \cdot \|T^{*k-2}\| \|T^*x\|$$
$$\leqq \|T\| \|T^*\|^{k-2} \|T^*x\| \leqq \|T^*x\|.$$

On the other hand

$$\|T^*x\| = \|T^{*k}x\| = \|T^*T^{*k-1}x\| = \|T^*T^{k-1}x\| \leqq \|T^*\| \|T^{k-2}\| \cdot \|Tx\|$$
$$\leqq \|T^*\| \|T\|^{k-2} \|Tx\| \leqq \|Tx\|$$

consequently we have $\|Tx\| = \|T^*x\|$, so T is normal.

Partially isometricity of T.

$$\|Tx - TT^*Tx\|^2 = \|Tx\|^2 - (Tx, TT^*Tx) - (TT^*Tx, Tx) + \|TT^*Tx\|^2$$
$$= \|Tx\|^2 - 2\|T^*Tx\|^2 + \|TT^*Tx\|^2$$

$$\begin{aligned}
&\leqq \|Tx\|^2-2\|T^*Tx\|^2+\|T^*Tx\|^2\\
&=\|Tx\|^2-\|T^*Tx\|^2=\|T^kx\|^2-\|T^*Tx\|^2\\
&=\|T^{k-1}Tx\|^2-\|T^*Tx\|^2\\
&=\|T^{*(k-2)}T^*Tx\|^2-\|T^*Tx\|^2\\
&\leqq \|T^*\|^{k-2}\|T^*Tx\|^2-\|T^*Tx\|^2\leqq 0
\end{aligned}$$

so we have $T=TT^*T$, therefore T is partially isometric.

Theorem 5. *If T is a normaloid and satisfies* (1), *then T is normal and partially isometric.*

Proof. If T is a normaloid, $\|T^k\|=\|T\|^k=\|T\|$, so we get $\|T\|\leqq 1$. By means of Theorem 4 so the proof is complete.

Corollary 4. *If T is paranormal and satisfies* (1), *then T is normal and partially isometric.*

The following well known theorem is also proved as a simple corollary of Theorem 4, we omit the proof.

Corollary 5. *If T is a contraction and satisfies the following condition.*

(2) $$T^k=I$$

then T is unitary.

By virtue of Theorem 3 and Corollary 5, we can easily deduce that *a periodic contraction is the direct sum of zero and a unitary operator.* This result is also derived from Theorem 4 (Problem 161 in [3]).

4. **Definition 3.** $\|T\|_N=\sup|(Tx, x)|$ for every unit vector x in H. $\|T\|_N$ is usually said to be the numerical radius of T ([3]). As a generalization of Corollary 5, we have the following theorem.

Theorem 6. *If T satisfies* (2) *and* $\|T\|_N\leqq 1$, *then T is unitary.*

Proof. $\sigma(T^k)=\sigma(T)^k=1$, so the spectrum of T lies in the unit circle, therefore $0\notin\sigma(T)$, there exists T^{-1}. By power inequality of $\|T\|_N$ and (2) we get

$$\|T^{-1}\|_N=\|T^{k-1}\|_N\leqq \|T\|_N^{k-1}\leqq 1.$$

Hence we have $\|T\|_N\leqq 1$ and $\|T^{-1}\|_N\leqq 1$, therefore T is unitary ([7]).

References

[1] T. Furuta: On the class of paranormal operators. Proc. Japan Acad., **43**, 594–598 (1967).
[2] T. Furuta, M. Horie, and R. Nakamoto: A remark on a class of operators. Proc. Japan Acad., **43**, 607–609 (1967).
[3] P. R. Halmos: Hilbert Space Problem Book. Van Nostrand, Princeton (1967).
[4] V. Istrătescu: On some hyponormal operator. Pacif. J. Math., **22**, 413–417 (1967).
[5] W. Mlak: Hyponormal contractions. Coll. Math., **18**, 137–142 (1967).
[6] J. G. Stampfli: Hyponormal operators. Pacif. J. Math., **12**, 1453–1458 (1962).
[7] ——: Normality and the numerical range of an operator. Bull. A. M. S., **72**, 1021–1022 (1966).
[8] O. Takenouchi: Kansûkaiseki Ensyû, Chap. II, Ex. 2-B, No. 27. Asakura, Tokyo (1968).