115. On the Schur Index of a Monomial Representation

By Toshihiko YAMADA
Department of Mathematics, Tokyo Metropolitan University

(Comm. by Kenjiro SHODA, M. J. A., Sept. 12, 1969)

In this note we give a method of determing the Schur index of a monomial representation of a finite group which is induced from a linear character of its normal subgroup. At the same time we obtain some other results which are useful in the theory of Schur index.

Notation and Terminology. G denotes a finite group whose unit element is 1. $|G|$ is the order of $G \cdot K$ is any given field of characteristic 0 and Ω the algebraic closure of K. An irreducible character χ of G always means an absolute one afforded by a representation of the group algebra ΩG over Ω. $m_K(\chi)$ is the Schur index of χ over K. $K(\chi)$ is the field obtained from K by adjunction of all values $\chi(g)$, $g \in G$. $\mathfrak{G}(K(\chi)/K)$ is the Galois group of $K(\chi)$ over K. For $\tau \in \mathfrak{G}(K(\chi)/K)$, χ^τ is the character of G defined by $\chi^\tau(g)=\chi(g)^\tau$. $e(\chi)=|G|^{-1}\chi(1)\sum_{g\in G}\chi(g^{-1})g$ is the minimal central idempotent of ΩG corresponding to χ. $a(\chi)=\sum_{\tau\in\mathfrak{G}(K(\chi)/K)} e(\chi^\tau)$ is the identity of the simple component A of KG with the property $\chi(A)\neq 0$ [2, V, 14. 12]. If H is a subgroup of G and ψ a character of H, ψ^G denotes the character of G induced from ψ. For a ring R and an integer n, R_n is the total matric algebra of degree n over R.

Lemma. *Let H be a subgroup of G and $Hg_1, \cdots, Hg_n$ all the distinct right cosets of H in G. Let ψ be an irreducible character of H such that ψ^G is irreducible. For simplicity, set $e_i=g_i^{-1}e(\psi)g_i$ $(i=1, \cdots \cdots, n)$. Then we have* (i) $e(\psi^G)=\sum_{i=1}^n e_i$, (ii) $e(\psi^G)\Omega G=e_1\Omega G+\cdots+e_n\Omega G$, (iii) $e_ie_j=0$ $(i\neq j)$, $e_ie_i=e_i$, $1\leq i, j\leq n$, (iv) $(\psi^\tau)^G=(\psi^G)^\tau$ *for any* $\tau\in\mathfrak{G}(K(\psi)/K)$.

Proof. (i) $e(\psi^G)=|G|^{-1}\psi^G(1)\sum_{g\in G}\psi^G(g^{-1})g=|H|^{-1}\psi(1)\sum_{g\in G}$

$$\sum_{i=1}^n \psi(g_ig^{-1}g_i^{-1})g=\sum_{i=1}^n g_i^{-1}\{|H|^{-1}\psi(1)\sum_{h\in H}\psi(h^{-1})h\}g_i=\sum_{i=1}^n e_i,$$

where $\psi(g)=0$ for $g\notin H$. (ii) It can be easily seen that $e(\psi)\Omega G\simeq e_i\Omega G$ $(i=1,\cdots,n)$ as right ΩG-modules and that $\dim_\Omega e(\psi)\Omega G=n\,\psi(1)^2$ and that $e(\psi^G)\Omega G\subset e_1\Omega G+\cdots+e_n\Omega G$. Hence, $(n\,\psi(1))^2=\dim_\Omega e(\psi^G)\Omega G \leq \dim_\Omega\{e_1\Omega G+\cdots+e_n\Omega G\}\leq n^2\psi(1)^2$. This proves (ii). (iii) We observe that $e_i=e(\psi^G)e_i=e_1e_i+\cdots+e_ie_i+\cdots+e_ne_i$. Since $e_1\Omega G+\cdots+e_n\Omega G$ is a direct sum, it follows that $e_ie_j=0$ $(i\neq j)$, $e_ie_i=e_i$.

(iv) $(\psi^{\tau})^G(g)=\sum_{j=1}^{n}\psi^{\tau}(g_j g g_j^{-1})=\left\{\sum_{j=1}^{n}\psi(g_j g g_j^{-1})\right\}^{\tau}=(\psi^G)^{\tau}(g),\ g\in G.$

Theorem 1. *Let H be a subgroup of G whose index in G is n. Let ψ be an irreducible character of H such that the induced character ψ^G is irreducible. Assume that $K(\psi)=K(\psi^G)$. If the simple component $a(\psi)KH$ of KH is isomorphic to D_r for a division algebra D over K and for an integer r, then the simple component $a(\psi^G)KG$ of KG is isomorphic to D_{rn}. In particular, $m_K(\psi^G)=m_K(\psi)$.*

Proof. Let $Hg_1, \cdots, Hg_n$ $(g_1=1)$ be all the distinct right cosets of H in G. From Lemma and the assumption $K(\psi)=K(\psi^G)$, it follows that $a(\psi^G)=\sum_{\tau\in\mathfrak{G}(K(\psi^G)/K)}e((\psi^G)^{\tau})=\sum_{\tau\in\mathfrak{G}(K(\psi)/K)}e((\psi^{\tau})^G)=\sum_{\tau}\sum_{i=1}^{n}g_i^{-1}e(\psi^{\tau})g_i=\sum_{i=1}^{n}g_i^{-1}a(\psi)g_i$. By Lemma, $g_i^{-1}e(\psi^{\tau})g_i\cdot g_j^{-1}e(\psi^{\tau})g_j=0\ (i\neq j)$. If $\tau,\tau'\in\mathfrak{G}(K(\psi^G)/K)$, $\tau\neq\tau'$, then $(\psi^{\tau})^G\neq(\psi^{\tau'})^G$, and so $e((\psi^{\tau})^G)\Omega G\cdot e((\psi^{\tau'})^G)\Omega G=0$. Hence, $g_i^{-1}e(\psi^{\tau})g_i\cdot g_j^{-1}e(\psi^{\tau'})g_j=0$. Thus, $g_i^{-1}a(\psi)g_i\cdot g_j^{-1}a(\psi)g_j=\sum_{\tau,\tau'}g_i^{-1}e(\psi^{\tau})g_i\cdot g_j^{-1}e(\psi^{\tau'})g_j=0$, and so $g_i^{-1}a(\psi)KHg_i\cdot g_j^{-1}a(\psi)KHg_j=0\ (i\neq j)$. Let $\delta\in a(\psi)KH$ be an idempotent of KH such that δKH is an irreducible right KH-module. Then the ring of KH-endomorphisms of δKH is isomorphic to the division algebra $\delta KH\delta$, which is anti-isomorphic to D. Denote by Ξ the ring of KG-endomorphisms of the right KG-module δKG. For $\xi\in\Xi$, $\xi(z)=\xi(\delta z)=\xi(\delta)z$, $z\in\delta KG$, where $\xi(\delta)\in\delta KG$. Hence for $\xi,\xi'\in\Xi$, $\xi=\xi'$ if and only if $\xi(\delta)=\xi'(\delta)$. Meanwhile, if $\xi(\delta)=\sum_{i=1}^{n}\delta s_i g_i, s_i\in KH$, then $\xi(\delta)=\xi(\delta^2)=\xi(\delta)\delta=\sum_{i=1}^{n}\delta s_i g_i\delta=\sum_{i=1}^{n}g_i\cdot g_i^{-1}\delta s_i g_i\cdot\delta=\delta s_1\delta\in\delta KH\delta$, because $g_i^{-1}\delta s_i g_i\in g_i^{-1}a(\psi)KHg_i$, $\delta\in a(\psi)KH$, and $g_i^{-1}a(\psi)KHg_i\cdot a(\psi)KH=0\ (i\neq 1)$. It follows readily that the ring Ξ is isomorphic to the division algebra $\delta KH\delta$, so that δKG is an irreducible right KG-module contained in $a(\psi)KG$. From the fact that $g_i^{-1}a(\psi)g_i$ $(i=1, \cdots, n)$ are orthogonal idempotents and $a(\psi^G)=\sum_{i=1}^{n}g_i^{-1}a(\psi)g_i$, it follows easily that $a(\psi^G)KG=g_1^{-1}a(\psi)g_1KG+\cdots+g_n^{-1}a(\psi)g_nKG$. Hence $a(\psi^G)KG$ contains δKG whose KG-endomorphism ring is anti-isomorphic to D. So we have $a(\psi^G)KG\simeq D_x$ for some integer x. As $g_i^{-1}a(\psi)g_iKG$ is isomorphic to $a(\psi)KG$ as right KG-modules and $\dim_K a(\psi)KG=n\cdot\dim_K a(\psi)KH=n\cdot\dim_K D_r=nr^2(D:K)$, we have $\dim_K a(\psi^G)KG=n^2r^2(D:K)$. Thus, $a(\psi^G)KG\simeq D_{rn}$.

Theorem 2. *Let H be a normal subgroup of G and ψ a linear character of H such that ψ^G is irreducible. Set $F=\{g\in G\,;\psi^g=\psi^{\tau(g)}$ for some $\tau(g)\in\mathfrak{G}(K(\psi)/K)\}$, where ψ^g is defined by $\psi^g(h)=\psi(ghg^{-1})$, $h\in H$. Let $Hf_1, \cdots, Hf_n$ be all the distinct cosets of H in F, and $f_if_j=h_{ij}f_{\nu(i,j)}$, $h_{ij}\in H$. Set $\tau(f_i)=\tau_i$ and $\beta(\tau_i,\tau_j)=\psi(h_{ij})$, $1\leq i,\ j\leq n$.*

Then we have (i) $F/H\simeq\{\tau_1, \cdots, \tau_n\}\simeq\mathfrak{G}(K(\psi)/K(\psi^F))$, (ii) β *is a factor set of* $\mathfrak{G}(K(\psi)/K(\psi^F))$ *consisting of roots of unity and the simple algebra* $a(\psi^F)KF$ *is isomorphic to the crossed product* $(\beta(\tau_i, \tau_j), K(\psi)/K(\psi^F))$, (iii) $m_K(\psi^G)=m_K(\psi^F)$. *In fact, if* $a(\psi^F)KF\simeq D_r$ *for a division algebra over* K *and for an integer* r, *then* $a(\psi^G)KG\simeq D_{rt}$, $t=(G:F)$.

Proof. As ψ^F is irreducible, $\psi^f=\psi$ if and only if $f\in H$. So the mapping: $f\mapsto\tau(f)$, $f\in F$ is a homomorphism from F into $\mathfrak{G}(K(\psi)/K)$ whose kernel is H. Hence $F/H\simeq\{\tau_1, \cdots, \tau_n\}$. For any $f\in F$, $(\psi^F)^{\tau_i}(f)=\sum_{j=1}^{n}\psi(f_j f f_j^{-1})^{\tau_i}=\sum_{j=1}^{n}\psi^{\tau_j\tau_i}(f)=\psi^F(f)$, and so $(\psi^F)^{\tau_i}=\psi^F$ $(i=1, \cdots, n)$. Conversely, if $\psi^F=(\psi^F)^\tau=(\psi^\tau)^F$ for some $\tau\in\mathfrak{G}(K(\psi)/K)$, there exists $f\in F$ such that $\psi^\tau=\psi^f$ [1, 45.6]. Hence τ is in $\{\tau_1, \cdots, \tau_n\}$. Therefore, $\mathfrak{G}(K(\psi)/K(\psi^F))\simeq\{\tau_1, \cdots, \tau_n\}$. Remark that $K(\psi)\supset K(\psi^F)\supset K(\psi^G)$. If $\psi^G=(\psi^G)^\tau=(\psi^\tau)^G$ for $\tau\in\mathfrak{G}(K(\psi)/K)$, there exists $g\in G$ such that $\psi^g=\psi^\tau$. Hence $g\in F$ and $\tau\in\mathfrak{G}(K(\psi)/K(\psi^F))$. This shows that $K(\psi^F)=K(\psi^G)$. Then the assertion (iii) is an immediate consequence of Theorem 1. If U is the matrix representation of F with the character ψ^F, $a(\psi^F)KF$ is isomorphic to the enveloping algebra $\mathrm{env}_K U=\{\sum_{f\in F}\alpha_f U(f)\,;\,\alpha_f\in K\}$ of U over K. For $h\in H$, $U(h)$ is the diagonal matrix $[\psi^{\tau_1}(h), \cdots, \psi^{\tau_n}(h)]$ with the diagonal elements $\psi^{\tau_1}(h), \cdots, \psi^{\tau_n}(h)$, and so $\mathrm{env}_K U'=\{[\theta^{\tau_1}, \cdots, \theta^{\tau_n}]\,;\,\theta\in K(\psi)\}\simeq K(\psi)$, where U' denotes the restriction of U to H. It is easily seen that $\mathrm{env}\, U=\sum_{i=1}^{n}\mathrm{env}\, U'\cdot U(f_i)$ and that the mapping $\tilde\tau_i\colon T\mapsto U(f_i)TU(f_i)^{-1}$, $T\in\mathrm{env}(U')$ is the automorphism of the field $\mathrm{env}\, U'$ corresponding to $\tau_i\in\mathfrak{G}(K(\psi)/K(\psi^F))$ and that $\{\tilde\tau_1, \cdots, \tilde\tau_n\}$ is the Galois group of the extension $\mathrm{env}\, U'/K(\psi^F)\cdot 1_n$, 1_n being the identity of Ω_n. Now it is well known that $K(\psi^F)\cdot 1_n$ is the center of $\mathrm{env}\, U$ and $(\mathrm{env}\, U : K(\psi^F)\cdot 1_n)=n^2$. Thus, we have the expression of $\mathrm{env}_K U$ as crossed product: $\mathrm{env}_K(U)=\sum_{i=1}^{n}\mathrm{env}_K U'\cdot U(f_i)=(\tilde\beta(\tilde\tau_i,\tilde\tau_j), \mathrm{env}_K U'/K(\psi^F)\cdot 1_n)$ with relations $U(f_i)TU(f_i)^{-1}=T^{\tilde\tau_i}$, $T\in\mathrm{env}_K U'$, $U(f_i)U(f_j)=U(h_{ij})U(f_{\nu(i,j)})$, $\tilde\beta(\tilde\tau_i, \tilde\tau_j)=U(h_{ij})$, $1\leq i, j\leq n$. Clearly, this crossed product is isomorphic to the crossed product $(\beta(\tau_i, \tau_j), K(\psi)/K(\psi^F))$.

As for the crossed product $A=(\beta(\tau_i, \tau_j), K(\psi)/K(\psi^F))$ in Theorem 2, we recall that if K is a finite extension of the rational p-adic number field Q_p for a prime p, then the $\mathfrak{P}$-invariant of A equals $\rho\cdot(K(\psi^F): Q_p(\psi^F))$ where ρ is the $\mathfrak{p}$-invariant of $B=(\beta(\tau_i, \tau_j), Q_p(\psi)/Q_p(\psi^F))$. Here $\mathfrak{P}$ and $\mathfrak{p}$ are the prime ideals of $K(\psi^F)$ and $Q_p(\psi^F)$ respectively, that divide p. Now B is a "Kreisalgebra" and its $\mathfrak{p}$-invariant was calculated by Witt [3].

References

[1] C. W. Curtis and I. Reiner: Repesentation Theory of Finite Groups and Associative Algebras. Interscience, New York (1962).
[2] B. Huppert: Endliche Gruppen. I. Springer, Berlin (1967).
[3] E. Witt: Die algebraische Struktur des Gruppenringes einer endlichen Gruppe über einem Zahlenkörper. J. reine angew. Math., **190**, 231–245 (1952).