9. *On a Class of Hypoelliptic Differential Operators*

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§1. Introduction. Let $A(x, y; \xi)$ and $B(x, y; \eta)$ be uniformly elliptic polynomials[1] in $\xi \in R^\nu$ and in $\eta \in R^\mu$, respectively, with coefficients in $C^\infty(\Omega)$ and $g(x)$ be a real valued function in $C^\infty(\Omega)$, not depending on y, where Ω is an open set of $R_x^\nu \times R_y^\mu$. In this paper, we consider the hypoellipticity[2] of linear partial differential operators of the form

(1) $$P = A(x, y; D_x) + g(x)^2 B(x, y; D_y),$$

where $D_x = (D_{x_1}, \cdots, D_{x_\nu})$ with $D_{x_j} = -i\partial/\partial x_j$ and $D_y = (D_{y_1}, \cdots, D_{y_\mu})$ with $D_{y_k} = -i\partial/\partial y_k$ $(i = \sqrt{-1})$. It is well known that if $g(x)$ vanishes at no point of Ω operator (1) is hypoelliptic in Ω. Indeed, we can immediately see that it is formally hypoelliptic there. For operator (1) in which $g(x)$ may vanish, we can prove

Theorem. *Suppose in operator* (1) *that A and B are uniformly elliptic in Ω and the coefficients of A are not dependent on the variable y and that there exists a multi-index $\alpha = (\alpha_1, \cdots \alpha_\nu) \in N^\nu$*[3] *such that $D_x^\alpha g = D_{x_1}^{\alpha_1} \cdots D_{x_\nu}^{\alpha_\nu} g$ vanishes at no point of Ω. Then the differential operator P of form* (1) *is hypoelliptic in Ω.*

This is motivated by the result of Dr. T. Matsuzawa (unpublished) that the operators on the (x, y)-plane: $D_x^{2l} + x^{2k} D_y^{2m}$ $(l, m = 1, 2, \cdots;$ $k = 0, 1, \cdots)$ are hypoelliptic in the plane (see [4]). One of the most important keys to the proof of Theorem is the inequality (H) which is stated in §2 and is one of the inequalities proved by Hörmander [2].

In §2 we prepare some lemmas and propositions, with the aid of which the proof of Theorem will be accomplished in §3.

§2. Preliminaries. Throughout this section we assume that A, B and g have the same meaning as in Theorem and that the degrees of A and B are $2l$ and $2m$ $(l, m = 1, 2, \cdots)$, respectively. First define norm $|||\cdot|||$ and its dual norm $|||\cdot|||'$ by

$$|||u|||^2 = \|D_x^l u\|^2 + \|g D_y^m u\|^2 + \|u\|^2, \quad |||v|||' = \sup_{u \in C_0^\infty(\Omega)} \frac{|\langle v, u\rangle|}{|||u|||},$$

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1) The $A(x, y; \xi)$ is called *uniformly elliptic* in ξ, if there exists a positive constant c such that $\operatorname{Re} A_0(x, y; \xi) \geq c|\xi|^{2l}$ for all $\xi \in R^\nu$ and all $(x, y) \in \Omega$ where $2l$ is the degree of A and A_0 denotes the leading part of A.

2) We say that P is *hypoelliptic* in Ω, if every $u \in \mathcal{D}'(\Omega)$ is infinitely differentiable in every open set where Pu is infinitely differentiable.

3) We denote by N the set of non-negative integers.

where norm $\|\cdot\|$ is the usual L^2-norm on Ω, $\langle v, u\rangle$ is the value of $v\in\mathcal{D}'(\Omega)$ at u, $\|D_x^l u\|^2=\sum_\alpha \|D_x^\alpha u\|^2$ $(\alpha\in N^\nu, |\alpha|=\alpha_1+\cdots+\alpha_\nu=l)$ and $\|D_y^m u\|^2=\sum_\beta \|D_y^\beta u\|^2$ $(\beta\in N^\mu, |\beta|=\beta_1+\cdots+\beta_\mu=m)$. Let $\varphi\in C^\infty(\bar\Omega)$. Clearly we have

$$(2)\qquad \begin{aligned} &|||\varphi v|||'\leq \text{const.}\,|||v|||',\\ &|||D_x^\alpha v|||'\leq \|v\| \qquad \text{for } |\alpha|\leq l\\ &|||gD_y^\beta v|||'\leq \|v\| \qquad \text{for } |\beta|\leq m.\end{aligned}$$

Let $L_\delta=1+\delta\left(\sum_{j=1}^{\nu} D_{x_j}^{2l}+\sum_{k=1}^{\mu} D_{y_k}^{2m}\right)$ for $\delta>0$, and defind $L_\delta^{-1}v$ as the inverse Fourier transform of $\left\{1+\delta\left(\sum_{j=1}^{\nu}\xi_j^{2l}+\sum_{k=1}^{\mu}\eta_k^{2m}\right)\right\}^{-1}\hat v(\xi,\eta)$. ($\hat v(\xi,\eta)$ is the Fourier transform of v). It then follows immediately that, for every $v\in L^2(\Omega)$, $\alpha\in N^\nu$ and $\beta\in N^\mu$,

$$(3)\qquad \|\delta^\theta D_x^\alpha D_y^\beta L_\delta^{-1}v\|\leq\|v\|,\qquad \text{if } \theta=\frac{|\alpha|}{2l}+\frac{|\beta|}{2m}\leq 1.$$

By using (3) we can verify

Lemma 1. *For any compact set $K\subset\Omega$, there exists a positive constant C*[4] *such that*

$$|||L^{-1}u|||\leq C|||u|||,\qquad u\in C_0^\infty(K),$$
$$|||L^{-1}f|||'\leq C|||f|||',\qquad f\in\mathcal{E}'_K.\text{[5]}$$

Lemma 2. *Let $l_1,\cdots,l_n$ be integers ≥ 1. Then it follows that the inverse Fourier transform $J(x)$ of $\left(1+\sum_{j=1}^{n}\xi_j^{2l_j}\right)^{-1}$ with respect to $\xi\in R^n$ is infinitely differentiable except at the origin and that there exists a positive constant a and a function $w(x)$ which is defined in $|x|>0$ and is bounded in $|x|>\varepsilon$ for every $\varepsilon>0$ as well as its derivatives, such that $J(x)=\exp(-a|x|)w(x)$ for $|x|>0$.*

The proof will be completed by induction on n.

Now we return to the operator P. Using the Gårding inequality, we can assert that, for every compact set $K\subset\Omega$,

$$(4)\qquad |||u|||\leq C(|||Pu|||'+\|u\|),\qquad u\in C_0^\infty(K).$$

If we habitually introduce norm $\|\cdot\|_s$ for real s which is defined by $\|u\|_s^2=\iint(1+|\xi|^2+|\eta|^2)^s|\hat u(\xi,\eta)|^2d\xi d\eta$, it then follows from the assumption on $g(x)$ that the inequality

$$(\mathrm{H})\qquad \|u\|_\varepsilon\leq C\left(\sum_{j=1}^{\nu}\|D_{x_j}u\|+\sum_{k=1}^{\mu}\|gD_{y_k}u\|+\|u\|\right),\qquad u\in C_0^\infty(K)$$

is valid for some positive number $\varepsilon\leq 1$ depending only on $g(x)$ and K (for the proof see [2]). From (H) we can derive

4) From now on, letters C, C' stand for positive constants depending on compact set in Ω.

5) By $\mathcal{E}'_K$ we denote the set of $u\in\mathcal{D}'(\Omega)$ with support K.

(5) $$\|u\|_\varepsilon \le C|||u|||, \qquad u \in C_0^\infty(K).$$

In the below we shall state three propositions.

Proposition 1. *The P is partially hypoelliptic in x, that is, every $u \in \mathcal{D}'(\Omega)$ is regular in x on every open set where Pu is regular in x (for the notion "regular in x" see Gårding-Malgrange* [1]).

Proof. We have only to recall that the P is just one of operators introduced by Mizohata [5] (cf. Kato [3]).

Proposition 2. *Let K be any compact set in Ω. Then it follows that every $v \in L^2(\Omega)$ such that $\|D_x^{l-1}v\| < \infty$, $\|gD_y^{m-1}v\| < \infty$, $|||Pv|||' < \infty$ and* supp $[v] \subset K$ *satisfies $|||v||| < \infty$ and belongs to H^ε*[6] *with some positive number $\varepsilon \le 1$ depending only on $g(x)$ and K.*

Proof. Let K_0 be a compact set in Ω. By U_{K_0} we denote the set of $u \in L^2(\Omega)$ such that $\|D_x^{2l}u\| < \infty$, $\|D_y^{2m}u\| < \infty$ and supp $[u] \subset K_0$. From (4) and (5) we can easily obtain the inequality

(6) $$\|u\|_\varepsilon \le C|||u||| \le C'(|||Pu|||' + \|u\|), \qquad u \in U_{K_0}.$$

Now let v be an element of $L^2(\Omega)$ such that supp $[v] \subset K$, $\|D_x^{l-1}v\| < \infty$, $\|gD_y^{m-1}v\| < \infty$ and $|||Pv|||' < \infty$. For $\delta > 0$, we set $v_\delta = \chi L_\delta^{-1}v$, where $\chi \in C_0^\infty(\Omega)$ such that $0 \le \chi \le 1$ and $\chi = 1$ in a neighborhood ω of K. Putting supp $[\chi] = K_0$, we have $v_\delta \in U_{K_0}$. So that (6) is valid for $u = v_\delta$. If we assume the existence of a number M such that

(7) $$|||Pv_\delta|||' \le M \qquad \text{for all } \delta > 0,$$

it then follows from (6) that $|||v||| < \infty$ and $\|v\|_\varepsilon < \infty$. Thus we have only to prove (7). In ω we have $L_\delta P v_\delta = Pv + [L_\delta, P]v_\delta$, provided $[X, Y] = XY - YX$. Accordingly,

$$Pv_\delta = L_\delta^{-1}Pv + L_\delta^{-1}[L_\delta, P]v_\delta + L_\delta^{-1}h_\delta,$$

where $h_\delta = 0$ in ω, supp $[h_\delta] \subset$ supp $[\chi]$ and $h_\delta = PL_\delta \chi L_\delta^{-1}v$ out of ω. By (2) and Lemma 1 we have $|||L_\delta^{-1}Pv|||' \le C|||Pv|||'$ and $|||L_\delta^{-1}h_\delta|||' \le C\|h_\delta\|$. If $E(x, y)$ denotes the inverse Fourier transform of $\left(1 + \sum_{j=1}^{\nu} \xi_j^{2l} + \sum_{k=1}^{\mu} \eta_k^{2m}\right)^{-1}$, we have

$$(L_\delta^{-1}v)(x, y) = \delta^{-1/2l - 1/2m} \iint E\left(\frac{x - x'}{\delta^{1/2l}}, \frac{y - y'}{\delta^{1/2m}}\right) v(x', y')dx'dy'.$$

With the aid of Lemma 2, we can assert that if $(x, y) \notin \omega$, any derivative of $L_\delta^{-1}v$ decreases faster than any power of δ when $\delta \to 0$. Hence we have $\|h_\delta\| \to 0$ as $\delta \to 0$. Finally we can deduce from (2) and (3)

$$|||L_\delta^{-1}[L_\delta, P]v_\delta|||' \le C(\|D_x^{l-1}v\| + \|gD_y^{m-1}v\| + \|v\|).$$

This completes the proof. Q.E.D.

We denote by $H(s, t)$ (s, t real) the set of $u \in \mathcal{S}'(R^{\nu+\mu})$ such that

$$\|u\|_{s,t}^2 = \iint (1 + |\xi|^2)^s (1 + |\eta|^2)^t |\hat{u}(\xi, \eta)|^2 d\xi d\eta < \infty$$

It is clear that $H^{2\varepsilon} \subset H(\varepsilon, \varepsilon)$ for $\varepsilon \ge 0$.

6) $H^\varepsilon = \{u \in \mathcal{S}'(R^{\nu+\mu});\ \|u\|_\varepsilon < \infty\}$.

Proposition 3. *If v is in $H(\varepsilon, \varepsilon)\cap \mathcal{E}'(\Omega)$ for a positive number $\varepsilon\leq 1$ and satisfies $|||v|||<\infty$, it then follows that*

$$(8)_x \qquad D_x^\alpha v \in H(\varepsilon/2^{|\alpha|}, \varepsilon/2^{|\alpha|}) \qquad \text{for } |\alpha|\leq l-1.$$

$$(8)_y \qquad D_y^\beta (gv) \in H(\varepsilon/2^{|\beta|}, \varepsilon/2^{|\beta|}) \qquad \text{for } |\beta|\leq m-1.$$

Proof. First of all we note that, for real θ,

$$(9) \qquad u\in H(\theta-1, \theta)\cap H(1,0) \Rightarrow u \in H(\theta/2, \theta/2).$$

Indeed, using the Schwarz inequality, we have

$$\|u\|^2_{\theta/2,\theta/2}=\iint (1+|\xi|^2)^{(\theta-1)/2}(1+|\eta|^2)^{\theta/2}|\hat{u}|(1+|\xi|^2)^{1/2}|\hat{u}|\,d\xi d\eta$$

$$\leq \left(\iint (1+|\xi|^2)^{\theta-1}(1+|\eta|^2)^{\theta}|\hat{u}|^2 d\xi d\eta\right)^{1/2}\left(\iint (1+|\xi|^2)|\hat{u}|^2 d\xi d\eta\right)^{1/2}.$$

We shall prove $(8)_x$ and $(8)_y$ by induction on $|\alpha|$ and $|\beta|$ respectively. It follows from the assumption on v that $(8)_x$ is valid for $|\alpha|=0$ and that $D_x^\alpha v\in L^2$ for $|\alpha|\leq l$. Hence we have $D_x^\alpha v \in H(1,0)$ for $|\alpha|\leq l-1$. Suppose that $(8)_x$ is valid for $|\alpha|\leq l-2$. Then

$$D_x^\alpha v\in H(\varepsilon/2^{|\alpha|-1}-1, \varepsilon/2^{|\alpha|-1})\cap H(1,0),$$

for $1\leq|\alpha|\leq l-1$. Using (9), we can conclude $(8)_x$. By the same argument as above we can assert $(8)_y$.

§3. Proof of Theorem. Let $u\in \mathcal{D}'(\Omega)$ such that $Pu\in C^\infty(\Omega)$ and Ω_0 be an arbitrary open subset of Ω such that $\bar{\Omega}_0\subset\Omega$. From Proposition 1 it follows that, for any integer $n\geq 0$ satisfying $2n\geq l-1$, there exists an integer $s\leq 0$ such that, for every $\varphi\in C_0^\infty(\Omega_0)$, $\varphi u\in H(2n, s)$, i.e., $u\in H^{2n,s}_{\text{loc}}(\Omega_0)$ (for the existence of such s, see [1]). Then, putting $t=s-m+1$, we have, for every $\psi\in C_0^\infty(\Omega_0)$,

$$(10)_t \qquad \begin{array}{ll} E_t D_x^\alpha(\psi u)\in L^2 & \text{for } |\alpha|\leq l-1, \\ E_t D_y^\beta(g\psi u)\in L^2 & \text{for } |\beta|\leq m-1, \end{array}$$

where $E_t v$ denotes the inverse Fourier transform of $(1+|\eta|^2)^{t/2}\hat{v}(\xi,\eta)$. Using $(10)_t$ we can derive from (2) the inequality $|||P(\varphi E_t(\psi u))|||'<\infty$ for all $\varphi, \psi\in C_0^\infty(\Omega_0)$. This, together with Propositions 2 and 3, quarantees $(10)_{t+d}$ with $d=\min(\varepsilon/2^l, \varepsilon/2^m)$. We can continue in this fashion and obtain $(10)_{2n-s}$. Thus we have $\psi u\in H(2n,s)\cap H(0,2n-s)$ for every $\psi\in C_0^\infty(\Omega_0)$, from which, by the similar argument as in (9), follows $u\in H^{n,n}_{\text{loc}}(\Omega_0)$. This completes the proof of Theorem, since n and Ω_0 are arbitrary.

References

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