28. *Axioms for Boolean Rings*

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G. R. Blakley, S. Ôhashi, K. Iséki and the author gave some new definitions of commutative rings and semirings (see [1]-[4]). K. Iséki gave some new axiom systems for Boolean rings (see [5]). In this note, we shall give other definitions of Boolean rings with unity.

Theorem 1. *A set with two nullary operations,* 0 *and* 1, *and with two binary operations,* + *and juxtaposition, such that*

1.1) $r+0=r$,

1.2) $r1=r$,

1.3) $(r+r)a=0$,

1.4) $(a+(br+cz))r=(br+ar)+z(cr)$

for any a, b, c, r, z, *is a Boolean ring with unity.*

Proof. We can prove this theorem as follows.

1.5) $r+r$

$=(r+r)1$ by 1.2.

$=0$ by 1.3.

1.6) $0a$

$=(0+0)a$ by 1.1.

$=0$ by 1.3.

1.7) $a+b=b+a$ (See 1.7 in [4])

1.8) $cz=zc$ (See 1.8 in [4])

1.9) $a+(b+c)=(a+b)+c$ (See 1.9 in [4])

1.10) $(zc)r=z(cr)$ (See 1.10 in [4])

1.11) $(a+c)r$

$=(a+(0r+c1))r$ by 1.6, 1.2, 1.7, 1.1.

$=(0r+ar)+1(cr)$ by 1.4.

$=ar+cr$ by 1.6, 1.7, 1.1, 1.8, 1.2.

1.12) rr

$=(0+(1r+00))r$ by 1.7, 1.8, 1.6, 1.2, 1.1.

$=(1r+0r)+0(0r)$ by 1.4.

$=r$ by 1.6, 1.1, 1.8, 1.2.

1.13) For given a, b, the equation $a+x=b$ is solvable. Let $x=a+b$. $a+(a+b)$

$=(a+a)+b$ by 1.9.

$=b$ by 1.5, 1.7, 1.1.

Hence $a+b$ is one solution of the equation.

Therefore the proof of Theorem 1 is complete.

Theorem 2. *A set with two nullary operations,* 0 *and* 1, *and with two binary operations,* + *and juxtaposition, such that*

2.1) $r+0=0+r=r$,

2.2) $(r+r)a=0$,

2.3) $(a+(br+cz))r+s=((br+ar)+z(cr))+s1$

for any a, b, c, r, z, *is a Boolean ring with unity.*

Proof. We can prove this theorem as follows.

2.4) $0a=0$ (See 1.6)

2.5) $s1$

$=((0r+0r)+0(0r))+s1$ by 2.4, 2.1.

$=(0+(0r+00))r+s$ by 2.3.

$=s$ by 2.4, 2.1.

2.6) $(a+(br+cz))r$

$=(br+ar)+z(cr)$ by 2.1, 2.3, 2.4.

The remaining part of the proof can be trivially given by using Theorem 1.

Theorem 3. *A set with two nullary operations,* 0 *and* 1, *and with two binary operations,* + *and juxtaposition, such that*

3.1) $r+0=0+r=r$,

3.2) $r1=r$,

3.3) $(a+(br+cz))r+(t+t)d=(br+ar)+z(cr)$

for any a, b, c, d, r, t, z, *is a Boolean ring with unity.*

Proof. We can prove this theorem as follows.

3.4) $(t+t)d$

$=(0+(01+01))1+(t+t)d$ by 3.2, 3.1.

$=(01+01)+1(01)$ by 3.3.

$=10$ by 3.2, 3.1.

3.5) 10

$=(0+0)1$ by 3.4.

$=0$ by 3.1, 3.2.

3.6) $(t+t)d$

$=0$ by 3.4, 3.5.

3.7) $(a+(br+cz))r$

$=(br+ar)+z(cr)$ by 3.3, 3.6, 3.1.

The remaining part of the proof can be trivially given by using Theorem 1.

Theorem 4. *A set with two nullary operations,* 0 *and* 1, *and with two binary operations,* + *and juxtaposition, such that*

4.1) $r+0=0+r=r$,

4.2) $01=0$,

4.3) $(a+(br+cz))r+(s+(t+t)d)=((br+ar)+z(cr))+s1$

for any a, b, c, d, r, s, t, z, *is a Boolean ring with unity.*

Proof. We can prove this theorem as follows.

4.4) $(t+t)d=10$ (See 3.4)

4.5) $10=0$ (See 3.5)

4.6) $(t+t)d=0$ (See 3.6)

4.7) $(a+(br+cz))r+s=((br+ar)+z(cr))+s1$ (See 3.7)

The remaining part of the proof can be trivially given by using Theorem 2.

References

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