26. *On Definition for Commutative Idempotent Semirings*

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Recently Professor S. Tamura (2) gave some new axioms for commutative rings and semirings. In this Note, we shall give some axiom systems for commutative idempotent semirings. By idempotent semirings, we mean that the addition and the multiplication are both idempotent. This class of semirings is very important in algebraic systems.

First of all, I give a remark. In my previous paper (1), the terminology "distributive lattice" in Theorem 2 and its proof should be replaced by "commutative idempotent semiring".

Let $\langle X, +, \cdot, 0, 1\rangle$ be an algebraic system, where 0 and 1 are elements of X, $+$ and $\cdot$ are binary operations on X. As in the previous paper (1), we denote $a\cdot b$ by ab.

Theorem 1. $\langle X, +, \cdot, 0, 1\rangle$ *is a commutative idempotent semiring, if and only if it satisfies the following conditions*:

1.1) $r+0=r$,

1.2) $r1=r$,

1.3) $0a=0$,

1.4) $((a+br)^{+}cz+d+d)r=br+(ar+z(cr)+dr)$

for every a, b, c, d, r, z.

Proof. It is obvious that every commutative idempotent semiring satisfies 1.1)–1.4). We shall prove the "if" part.

1.5)	$a+b=((a+b1)+00+0+0)1$	{2, 3, 1}
	$=b1+(a1+0(01)+01)$	{4}
	$=b+a.$	{2, 3, 1}
1.6)	$cz=((0+01)+cz+0+0)1$	{1, 5, 3, 2}
	$=01+(01+z(c1)+01)$	{4}
	$=zc.$	{3, 1, 5, 2}
1.7)	$(b+a)+c=(a+b)+c$	{5}
	$=((a+b1)+c1+0+0)1$	{2, 1}
	$=b1+(a1+1(c1)+01)$	{4}
	$=b+(a+c).$	{2, 6, 3, 1}
1.8)	$(cz)r=((0+0r)+zc+0+0)r$	{6, 1, 3, 5}
	$=0r+(0r+c(zr)+0r)$	{4}
	$=c(zr).$	{3, 1, 5}
1.9)	$(a+c)r=((a+0r)+c1+0+0)r$	{3, 2, 1, 5}

$=0r+(ar+1(cr)+0r)$ {4}

$=ar+cr.$ {3, 1, 5, 2, 6}

1.10) $d+d=((0+01)+00+d+d)1$ {1, 3, 5, 2}

$=01+(01+0(01)+d1)$ {4}

$=d.$ {3, 1, 5, 2}

1.11) $r^2=((0+1r)+00+0+0)r$ {2, 6, 1}

$=1r+(0r+0(0r)+0r)$ {4}

$=r.$ {3, 1, 2, 6}

Therefore a set X is a commutative idempotent semiring.

Theorem 2. *$\langle X, +, \cdot, 0, 1\rangle$ is a commutative idempotent semiring, if and only if it satisfies the following conditions:*

2.1) $r+0=0+r=r,$

2.2) $0a=0,$

2.3) $((a+br)+cz+d+d)r+s$

$=br+(ar+z(cr)+dr)+s1$

for every a, b, c, d, r, s, z.

Proof. The "only if" part is obvious. The following is the proof of "if" part.

2.4) $s1=0r+(0r+0(0r)+0r)+s1$ {2, 1}

$=((0+0r)+00+0+0)r+s$ {3}

$=s.$ {2, 1}

2.5) $((a+br)+cz+d+d)r$

$=((a+br)+cz+d+d)r+0$ {1}

$=br+(ar+z(cr)+dr)+01$ {3}

$=br+(ar+z(cz)+dr).$ {4, 1}

Therefore Theorem 2 follows from Theorem 1.

Theorem 3. *$\langle X, +, \cdot, 0, 1\rangle$ is a commutative idempotent semiring, if and only if the following conditions hold:*

3.1) $r+0=0+r=r,$

3.2) $r1=r,$

3.3) $0e+((a+br)+cz+d+d)r$

$=br+(ar+z(cr)+dr)$

for every a, b, c, d, e, r, z.

Proof. We shall prove only the "if" part.

3.4) $0e=0e+((0+01)+01+0+0)1$ {2, 1}

$=01+(01+1(01)+01)$ {3}

$=0+(0+10).$ {2, 1}

3.5) $0+(0+10)=01$ {4}

$=0.$ {2}

3.6) $((a+br)+cz+d+d)r$

$=0e+((a+br)+cz+d+d)r$ {4, 5, 1}

$=br+(ar+z(cr)+dr).$ {3}

Therefore Theorem 3 follows from Theorem 1.

Theorem 4. $\langle X, +, \cdot, 0, 1\rangle$ *is a commutative idempotent semiring, if and only if it satisfies the following conditions*:

4.1) $r+0=0+r=r$,

4.2) $01=0$,

4.3) $0e+((a+br)+cz+d+d)r+s$
$=br+(ar+z(cr)+dr)+s1$

for every a, b, c, d, e, r, s, z.

Proof. The proof of the "if" part is reduced to Theorem 2.

4.4) $0e=0e+((0+01)+01+0+0)1+0$ $\{1, 2\}$
$=01+(01+1(01)+01)+01$ $\{3\}$
$=10.$ $\{2, 1\}$

4.5) $10=01$ $\{4\}$
$=0.$ $\{2\}$

4.6) $((a+br)+cz+d+d)r+s$
$=0e+((a+br)+cz+d+d)r+s$ $\{4, 5, 1\}$
$=br+(ar+z(cr)+dr)+s1.$ $\{3\}$

Therefore we complete the proof of Theorem 4.

References

[1] S. Ôhashi: On definitions of Boolean rings and distributive lattices. Proc. Japan Acad., **44**, 1015–1017 (1968).

[2] S. Tamura: Axioms for commutative rings. Proc. Japan Acad., **46**, 116–120 (1970).