24. *Characteristic Pseudo Quasi Topological Spaces*

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Introduction. One defines a characteristic pseudo quasi metric spaces as the topological space generated by a pseudo quasi metric function whose range is $\{0,1\}$. Since every finite topological space is a special case of the characteristic pseudo quasi spaces, many results concerning finite topological spaces which have been known by precedents ([2], [12], [13]) are considered as the corollaries of the results of characteristic pseudo quasi metric spaces. Furthermore, every pseudo quasi metric is considered as a transformation into the reals by $f_x(y)=d(x,y)$ for each $x \in X$ and one induces an equivalent matrix representation for a finite topological space and the algebraic structure of the matrix representation is studied. Similarly, it is observed that these functions induce partial ordered relation on X.

1. This chapter is mainly concerned with necessary definitions and theorems which will be used for the discussion of the later chapters.

1.1. Definition. A *p.q.* (pseudo quasi) metric (see [6]) "d" is said to be characteristic *p.q.* (or *c.p.q.*) metric iff whose range is $\{0,1\}$.

One observes *c.p.q.* metrics act like a characteristic function on the minimum base for each $x \in X$.

For each *c.p.q.* metrix $\tilde{d}$, there exists the conjugate *c.p.q.* metric $\underset{\sim}{d}$, which is defined as $\underset{\sim}{d}(x,y)=\tilde{d}(y,x)$.

Notation. (1) $\tilde{S}(x,\varepsilon)=\{y : \tilde{d}(x,y)<\varepsilon, \varepsilon>0\}$

(2) $\underset{\sim}{S}(x,\varepsilon)=\{y : \tilde{d}(x,y)<\varepsilon, \varepsilon>0\}$

1.2. Definition. Let $\tilde{C}$ be the topology whose base is $\{\tilde{S}(x,\varepsilon)\}$ and it is said to be the characteristic topology of $\tilde{d}$. Similarly, $\underset{\sim}{C}$ is defined and $(X,\tilde{C},\underset{\sim}{C})$ is called the *c.p.q.* bitopological space.

The following theorem is well known ([4]-[6], [9])

1.3. Theorem. *Let the notation "$A \Rightarrow B$" be A implies B.*

$$p.q.\ \textit{bitopology}\begin{cases} \Rightarrow \textit{p-perfectly normal} \Rightarrow \textit{p-completely normal} \\ \quad \Rightarrow \textit{p-normal} \\ \Rightarrow \textit{p-completely regular} \Rightarrow \textit{p-regular.} \end{cases}$$

where "p-" denote pairwise (e.g. p-regular stands for pairwise regular)

1.4. Theorem. *Let $(X,\tilde{C},\underset{\sim}{C})$ be a c.p.q. bitopological space.*

$$U \in \tilde{C} \text{ iff } U^c \in \underset{\sim}{C}.$$

Proof. If $U \in \tilde{C}$, then $U = \bigcup_{x \in U} S(x, 1)$, where $S(x, 1) = \{y : \tilde{d}(x, y) = 0\}$. Therefore, for $x \in U$ and $z \in U^c$, $\tilde{d}(x, z) = 1$ and $\underset{\sim}{d}(z, x) = 1$. Consequently, $\underset{\sim}{S}(z, 1) \cap U = 0$ for every $z \in U^c$ and U is $\underset{\sim}{C}$-closed.

Similarly, the rest of the proof is induced.

2. It is easy to see that every *c.p.q.* topology has the minimum base for each $x \in X$. In particular, every finite topological space is a *c.p.q.* space.

2.1. Theorem. *Let $(X, \tilde{C}, \underset{\sim}{C})$ be a c.p.q. bitopological space.*

$$\tilde{C} = \underset{\sim}{C} \text{ iff } \tilde{C} \text{ is } R_0.$$

Proof. Assume $x \in U \in \tilde{C}$. Since $\tilde{C}$ is R_0, $\bar{x} \in U$ where $\bar{x}$ is $\tilde{C}$ closure of x.

Therefore, for any $y \in U$ $\tilde{d}(y, x) = 1$ and $\tilde{S}(y, 1) \cap U = \emptyset$.

Consequently, U is $\tilde{C}$-closed and apply (1.5) and the proof is completed.

However, if R_0 is replaced by normality, then a *c.p.q.* bitopological space $(x, \tilde{C}, \underset{\sim}{C})$ does not imply $\tilde{C} = \underset{\sim}{C}$.

2.2. Example. Let $X = \{x_1, x_2, x_3\}$ and $\tilde{C} = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_2\}\{x_3\}, X, \emptyset\}$ then it is a normal space but it is not a regular space.

2.3. Theorem. *If $(X, \tilde{C}, \underset{\sim}{C})$ is a c.p.q. bitopological space, then $(X, \tilde{C})$ is connected iff (X, C) is connected.*

Proof. apply (1.5).

2.4. Theorem. *Let (X, C) be a c.p.q. topological space. The following are equivalent*

(a) *(X, C) is pseudo metrisable.*

(b) *(X, C) is regular.*

(c) *(X, C) is completely regular.*

(d) *(X, C) is 0-dimensional.*

(e) *(X, C) is R_1.*

(f) *(X, C) is R_0.*

Proof. See (1.4), (1.5), (1.6) and (2.1).

Fletcher [2] obtained some parts of the above results in the case of X as a finite topological space by the use of quasi uniformity.

3. As a special case of *c.p.q.* metric topology, let X be a finite set. We study a matrix representation of $(X, \tilde{C})$ which is naturally induced by "$\tilde{d}$".

Let $X = \{x_1, x_2, \cdots, x_n\}$. Then with each finite set X, we can associate a column vector

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \sim x$$

and we know $\tilde{d}(x_j, x_i) = 0$ or 1 for each i and j. Therefore it is natural to make the following associations.

$$(x,\tilde{C})\sim\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tilde{C}\right)\sim\begin{pmatrix} 0 & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}=M_{(\tilde{C})}$$

(or the representation matrix of $\tilde{C}$)

where $a_{ij}=d(x_i, x_j)$.

Let R_i be the i^{th} row and C_j be the j^{th} column of $M_{(\tilde{C})}$. Then $M_{(\tilde{C})}=\begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}=(C_1, \cdots, C_n)$ and R_i is identified as $S_{\tilde{d}}(x_i, 1)$. R_i is said to be the vector representation of $S_{\tilde{d}}(x, 1)$ and let $B(x_i)$ be $R_i=(a_{i1}\cdots \overset{(i)}{0}\cdots a_{in})$, the smallest $\tilde{C}$-open set containing x_i.

Similarly, we can easily obtain $M_{(\underset{\sim}{C})}$ where $M_{(\underset{\sim}{C})}$ is the matrix representation of $\underset{\sim}{C}$ which is induced "$\underset{\sim}{d}$", the conjugate *c.p.q.* metric of "$\tilde{d}$".

With each x_i we associate

$$R_i^c=(a_{i1}, \cdots, \overset{(i)}{1}, a_{i(i-1)}, \cdots, a_{in})$$

$$\text{for } B(x_i)^c, \text{ where } a_{ij}=\begin{cases}1 \text{ if } \tilde{d}(x_j, x_i)=1, \\ 0 \text{ otherwise}\end{cases}$$

and let $(1, \cdots, 1)=\emptyset$.

Here, we need a suitable vector operation which is applicable in the new system.

3.1. Definition. (i) $R_i\cap R_j=((a_{i1}\cap a_{j1}), \cdots (a_{in}\cap a_{jn}))$
$=(\delta_1, \cdots, \delta_n)$

where
$$\delta_K=\begin{cases}0 \text{ if } a_{ik}+a_{jk}=0, \\ 1 \text{ if } a_{ik}+a_{jk}\geq 1\end{cases}$$

(ii) $R_i\cup R_j=((a_{i1}\cdot a_{j1}), \cdots, (a_{in}\cdot a_{jn}))$.

Then it is obvious that

(i) $R_i\cap R_j$ is the vector representation of the intersection of the x_i-nbhd and the x_j-nbhd in the characteristic base.

(ii) $R_i\cup R_j$ is the vector representation of the union of x_i-nbhd and x_j-nbhd and it will be identified an element of L.

Considering the fact that any intersection of two open sets is an open set or $\emptyset$ and the definition of $\cap$, we have the following:

3.2. Lemma. (I) *$R_i\cap R_i'-R_i''$, where R_i, R_i' and R_i'' are vector representation on x_i nbhd of different c.p.q. topologies.*

(II) $(R_i\cap R_i')\cap R''=R_i\cap(R_i\cap R'')$,

(III) $R_i\cap R_i'=R_i'\cap R_i$,

(IV) $(R_i\cap R_0)=R_i$, *where* $R_0=(0, \cdots, 0)$.

Let $\mathfrak{m}=\{M_{(c)}\}$ be the set of all the matrix representations of C_i, where C_i is a topology defined on X (X is a finite set). A matrix operation on $\mathfrak{m}$ is defined as follows:

$$M_{(C)}\cap M_{(C')}=\begin{pmatrix}0 & & \\ & \ddots & \\ a_{ij} & & 0\end{pmatrix}\cap\begin{pmatrix}0 & & \\ & \ddots & \\ a'_{ij} & & 0\end{pmatrix}$$

$$=\begin{pmatrix}R_1\\ \vdots\\ R_n\end{pmatrix}\cap\begin{pmatrix}R'_1\\ \vdots\\ R'_n\end{pmatrix}=\begin{pmatrix}R_1 & \cap & R'_1\\ \vdots & & \vdots\\ R'_n & \cap & R_n\end{pmatrix}$$

Since the intersection of two characteristic basis is a characteristic base and

$$M_0=\begin{pmatrix}0\cdots 0\\ \cdots\cdots\\ 0\cdots 0\end{pmatrix}$$

is the matrix representation of indiscrete topology on X and $M_0\in\mathfrak{m}$. Furthermore, M_0 is considered as the identity element under the operation $\cap$. Similarly, we can show $\mathfrak{m}$ has the same structure with respect to $\cap$.

Therefore

3.3. Theorem. *$\mathfrak{m}$ is a commutative monid structure with respect to "$\cap$" and "$\cup$" operation.*

Here we discuss a few properties of the finite topological properties with, the matrix representation.

3.4. Definition. A bitopological space (X, L_1, L_2) is $p-T_{1\frac{1}{2}}$ iff for every $x, y\in X, x\neq y$ there exist disjoint an L_i-nbhd of x and an L_j-nbhd of x, where $i=1$ or 2 and $i\neq j$ ([6]).

3.5. Lemma. *Let (X, L_1, L_2) be a p-regular space. It is $p-T_{1\frac{1}{2}}$ iff $L_i, i=1$ or 2 is T_0 topology.*

Proof is omitted.

3.6. Theorem. *Let $(X,\tilde{C})$ be T_0. If $M_{(\tilde{C})}=\begin{pmatrix}R_1\\ \vdots\\ R_n\end{pmatrix}$ and $M^T_{(\tilde{C})}=\begin{pmatrix}R^t_1\\ \vdots\\ R^t_n\end{pmatrix}$, then $R_i\cap R^t_j=(1,1,\cdots,1)$. $i\neq j$ for $i=1,\cdots,n$.*

Proof. (X, C) being to imply that $(X,\tilde{C},\underset{\sim}{C})$ is $p-T_{1\frac{1}{2}}$, by (3.5) for each $x_i, x_j\in X, x_i\neq x_j$ these nbhds have empty intersection.

3.7. Theorem. *Let (X, C) be T_1 and $M_{(C)}=\begin{pmatrix}R_1\\ \vdots\\ R_n\end{pmatrix}$. Then*

$$R_i=(1,\cdots,1,\overset{(i)}{0},1\cdots),\ for\ i=1,\cdots,n.$$

3.8. Theorem. *Let (X, L) be R_0. Then $M_{(\tilde{C})}=M_{(\underset{\sim}{C})}$.*

Proof. See (2.1) and (2.4).

By (3.6) and (3.8)

3.9. Corollary. *If (K, C) is T_0 and R_0, then it is T_1.* Obviously, the weight of (X, C) is the number of distinct row vectors of $M_{(C)}$ added by 1 for empty element $\emptyset$. Therefore

3.10. Theorem. *The weight of (X, C) does not exceed $(n+1)$ (See [12]).*

(3.6) and the definition immediately lead to:

3.11. Corollary. *If (X, C) is T_0 (See p. 1966 of [13]) then the weight is $(n+1)$.*

The followings are obvious from the definition of $M_{(C)}$

3.12. Theorem. *$(X, C_1) \supset (X, C_2)$ iff $a_{ij}^{C_1} = a_{ij}^{C_2}$ or $a_{ij}^{C_2} = 0$, where $a_{ij}^{C_n}$ is the element of i^{th} row and j^{th} column of $M_{(C_n)}$, $n=1, 2$.*

3.13. Theorem. *(X, C) is connected iff $R_1 \cap R_j = (0, \cdots, 0)$ implies $R_1 \cap R_j \neq (1, \cdots, 1)$. For any $i, j = 1, \cdots n$.*

4. Stong [13] defined a partial ordering on a finite set. One generalizes the ordering in *c.p.q.* spaces.

4.1. Lemma. *A c.p.q. metric on X induces a partial order on X.*

Proof. Let "$>$" be defined as $x \geq y$ iff $d(x, y) = 0$. The following is true.

(i) $x \geq x$ (reflexive),

(ii) If $x \geq y$ and $y \geq z$, then $x \geq z$. Because by the triangular inequality $d(x, z) \leq d(x, y) + d(y, z)$ the right hand side for inequality is zero ($x \geq y$ and $y \geq z$ implies $d(x, y) = 0$ and $d(y, z) = 0$)

Therefore $d(x, z) = 0$ and $x \geq z$.

Similarly, the following is true

4.2. Lemma. *Let $(x, \geq)$ be a partial ordered set. "$\geq$" induces a c.p.q. metric.*

Proof. Let define $d: X \times X \to \{0, 1\}$ as

$$d(x, y) = 0 \text{ if } x \geq y,$$
$$d(x, y) = 1 \text{ otherwise.}$$

We obtain the following

(i) $d(x, x) = 0$ for all $x \in X$ and $d(x, y) \geq 0$.

(ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, x \in X$.

Because if $d(x, y) = 0$, then the result is trivial. Let's consider the case $d(x, y) = 1$. If $d(x, z) = d(z, y) = 0$ then $x \leq z$ and $z \leq y$, which implies $x \leq y$ and $d(x, y) = 0$. Therefore, at least either $d(x, z)$ or $d(z, y)$ is equal to 1 and the triangular inequality is held.

4.3. Theorem. *$f: (X, \tilde{C}) \to (X, \tilde{C})$ is continuous iff $f: (X, \geq) \to (X', \geq')$ is increasing.*

Proof. Let f be a continuous function at a and

$$U_{f(a)} = \{y\,;\, \tilde{d}'(f(a), y) < 1 \text{ or } \tilde{d}'(f(a), y) = 0\}.$$

Since $f^{-1}(U_{f(a)}) \supset \tilde{S}(a, 1)$ and $y \in U_{f(a)}$ which implies $f(a) \geq y$, if $b \in f^{-1}(y)$, then $b \in S(a, 1)$ and $a \geq b$.

The converse of the proof is easy.

Similarly,

4.4. Corollary. *$f:(X, C) \to (X', C')$ is continuous iff $f:(X, \geq)$ $(X', \geq')$ is decreasing.*

References

[1] A. Csazar: Foundation of General Topology. Macmillan, New York (1963).
[2] P. Fletcher: Finite topological spaces and quasi-uniform structures (to appear).
[3] J. L. Kelley: General Topology. Van Nostrand, Princeton, N. J. (1955).
[4] ——: Bitopological spaces. Proc. London Math. Soc., **13**(3), 71–89 (1963).
[5] E. P. Lane: Bitopological spaces and quasi uniform spaces. Proc. London Math. Soc., **17**, 241–256 (1967).
[6] Y.-W. Kim: Pseudo quasi metric spaces. Proc. Japan Acad., **44** (10), 1009–1012 (1968).
[7] V. Krishnamurthy: On the number of topologies on a finite set. Amer. Math. Monthly, **73**, 154–157 (1966).
[8] M. G. Murdeshwar and S. A. Naimpally: Quasi-uniform Topological Spaces. P. Noordroof, Groningen (1966).
[9] C. W. Patty: Bitopological spaces. Duke J. of Math., **34**, 387–391 (1967).
[10] W. J. Pervin: Quasi-uniformisation of topological spaces. Math. Ann., **147**, 316–317 (1962).
[11] ——: Foundation of General Topology. Academic Press, New York (1964).
[12] H. Sharp: Quasi-orderings and topologies on finite sets. Proc. Amer. Math. Soc., **17**, 1344–1349 (1966).
[13] R. E. Stong: Finite topological spaces. Trans. Amer. Math. Soc., **123**, 325–340 (1966).