59. Notes on Regular Semigroups

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In this note we shall give ideal-theoretical characterizations of regular semigroups whose left and/or right ideals are two-sided. Some ideal-theoretical characterizations of the class of regular semigroups were given in the author's recent paper [3].

For the notation and terminology we refer to A. H. Clifford and G. B. Preston's book [1].

Theorem 1. *For a semigroup S the following conditions are pairwise equivalent.*

(1) *S is a regular semigroup whose left ideals are two-sided.*

(2) *$B \cap L = BL$ for every bi-ideal B and every left ideal L of S.*

(3) *$L \cap Q = QL$ for each left ideal L and each quasi-ideal Q of S.*

Proof. (1) implies (2). Suppose that S is a regular semigroup whose left ideals are two-sided. Then by a recent result of the author [2] every bi-ideal B of S may be represented in the form

$$B = RI,$$

where R is a suitable right ideal and I is a suitable two-sided ideal of S. Next applying the well known regularity criterion due to L. Kovács and K. Iséki (see [1], p. 34) we obtain

$$B \cap L = RI \cap L = RIL = BL$$

for every bi-ideal B and every left ideal L of S.

(2) implies (3). This is evident because every quasi-ideal of an arbitrary semigroup S is a bi-ideal of S.

(3) implies (1). Let S be a semigroup with property (3). Then in case $Q = R$, R is an arbitrary right ideal of S, (3) implies that S is regular. Secondly in case $L = S$, $Q = L$, L is an arbitrary left ideal of S, condition (3) implies

$$L = L \cap S = LS,$$

that is, any left ideal L is also a right ideal of S.

The proof of our Theorem 1 is complete.

We state the left-right dual of Theorem 1.

Theorem 2. *For a semigroup S the following assertions are mutually equivalent.*

(4) *S is regular and each right ideal of S is two-sided.*

(5) *$B \cap R = RB$ for any bi-ideal B and for any right ideal R of S.*

(6) *$Q \cap R = RQ$ for every right ideal R and every quasi-ideal Q*

of S.

Next we formulate a criterion for a semigroup to be a semilattice of groups. This is a sharpening of an earlier criterion due to A. H. Clifford and G. B. Preston [1] and its proof is quite similar to that of Theorem 2 in the author's paper [4], and we omit it.

Theorem 3. *A semigroup S is a semilattice of groups if and only if S is regular and every one-sided ideal of S is two-sided.*

Finally Theorem 1, Theorem 2, and Theorem 3 imply the following result.

Theorem 4. *An arbitrary semigroup S is a semilattice of groups if and only if it satisfies both the conditions* (i) *and* (j) *of Theorem* 1 *and Theorem* 2, *where* $i=1, 2,$ *or* 3 *and* $j=4, 5,$ *or* 6.

References

[1] A. H. Clifford and G. B. Preston: The Algebraic Theory of Semigroups, Vol. I. Amer. Math. Soc., Providence, R. I. (1961).
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[3] ——: On characterization of regular semigroups. Proc. Japan Acad., **44**, 325–326 (1968).
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[5] ——: On semilattices of groups. Proc. Japan Acad., **45**, 383–384 (1969).