51. *A Generalization of the Riesz-Schauder Theory*

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(Comm. by Kunihiko KODAIRA, M. J. A., March 12, 1970)

We prove the following:

Theorem. *Let S be an analytic space and let $s \to K(s)$ be an analytic map of S into the ring of compact operators on a Banach space X. Then those points s of S for which $I+K(s)$ are not invertible form an analytic set in S.*

This is a generalization of the following assertion, which is a part of the Riesz-Schauder theory.

Corollary 1. *The spectrum of a compact operator is discrete.*

Proof. We apply the theorem to $I+sK$ and find that those s for which $I+sK$ are non-invertible form an analytic set in the complex plane $\boldsymbol{C}$, namely, discrete set of points or $\boldsymbol{C}$ itself. Because $I+sK$ is invertible when $s=0$, the latter case does not occur.

In the same way we can prove the following proposition which has applications in scattering theory.

Corollary 2. *Let $K(s)$ be a family of compact operators depending analytically on a parameter s in an open subset U of the complex plane $\boldsymbol{C}$. Then the set of all s for which $I+K(s)$ are non-invertible is either equal to U itself, or discrete in U.*

Proof of the Theorem.

We use a method given by Donin [1].

Since the concept of analytic subset is local, it suffices to consider a neighborhood of a fixed point $s_0 \in S$. Let N_0 and R_0 be the kernel and the range, respectively, of the map $I+K(s_0): X \to X$. Since $K(s_0)$ is compact, N_0 is of finite dimension, R_0 is of finite co-dimension, and therefore both are topological direct summands.

Let $X = N_0 \oplus Y$ and let P_0 be a continuous projection to R_0. Then the map $Y(s) = P_0 \circ [I+K(s)]|_Y : Y \to R_0$ gives, for $s=s_0$, an isomorphism $Y \cong R_0$. Since $Y(s)$ is continuous in s, $Y(s)$ is invertible for s sufficiently close to s_0. So, we can construct a map $h(s): N_0 \oplus R_0 \to X$ which is defined by $h(s)(y,z) = \{I - Y(s)^{-1} \circ P_0 \circ (I+K(s))\}y + Y(s)^{-1}z$, where $(y,z) \in N_0 \oplus R_0$. When $s=s_0$, this is an isomorphism $N_0 \oplus R_0 \cong X$, so $h(s)$ is an isomorphism for any s in some neighborhood of s_0, and we have, for s sufficiently near s_0, $\dim \ker (I+K(s)) = \dim \ker \{(I+K(s)) \circ h(s)\}$. On the other hand, we can show that $\ker \{(I+K(s) \circ h(s)\} \subset N_0$. In fact, for $(y,z) \in N_0 \oplus R_0$,

$$(I+K(s))\circ h(s)(y,z)$$
$$=(I+K(s))y-(I+K(s))Y(s)^{-1}P_0(I+K(s))y+(I+K(s))Y(s)^{-1}\cdot z$$
$$=(I+K(s))y-(P_0+I-P_0)\{(I+K(s))Y(s)^{-1}P_0(I+K(s))y\}$$
$$+(P_0+I-P_0)\{(I+K(s))Y(s)^{-1}z\}.$$

Here, by the definition of $Y(s)$, we have $P_0(I+K(s))Y(s)^{-1}P_0=P_0$. So this becomes,

$$=(I-P_0)\{(I+K(s))y-(I+K(s))Y(s)^{-1}P_0(I+K(s))y$$
$$+(I+K(s))Y(s)^{-1}(z)\}+P_0z$$
$$=(I-P_0)A+P_0z,$$

where, the last equality is the definition of the notation. Thus $(I+K(s))\circ h(s)(y,z)=0$ is equivalent to $(I-P_0)A=0$ and $P_0z=0$, because these are direct sums. In particular we have $P_0z=z=0$ since $z\in R_0$. This implies $\ker(I+K(s))\circ h(s)\subset N_0$. So we only have to study those s for which $(I+K(s))\circ h(s): N_0\to X$ has a non trivial kernel. Now that we have reduced the problem to the study of the maps from a space of finite dimension to X, the following lemma completes the proof of our theorem.

Lemma. *Consider an analytic family of linear maps $T(s): N_0\to X$ from a linear space N_0 of finite dimension to a Banach space X. Those s for which the ranks of the maps $T(s)$ are less than* $\dim N_0$ *form an analytic set in the parameter space.*

Proof. Let $P: X\to C^n$ ($n=\dim N_0<\infty$) be any projection of X to a subspace of finite dimension. $P\circ T(s)$ is a finite matrix, so the determinant of $P\circ T(s)$ is well defined. Taking as P all such projections, we have obviously

$$\{s\,;\,\mathrm{rank}\,T(s)<n\}=\bigcap_P \{s\,;\,\det(P\circ T(s))\}=0.$$

The right side is an analytic subset by the well-known theorem of Noether. This establishes the assertion.

If we make use of the k-th minors of $P\circ T(s)$, we have:

Corollary 3. *Those s, for which the dimensions of the kernel spaces of $I+K(s)$ are greater than k, form an analytic subset. Letting k run from* 0 *to ∞, we obtain a decreasing sequence of analytic subsets.* When we apply this corollary to the resolvents of a family of elliptic operators, we obtain an intuitive proof of the fact that the dimension of eigenspaces is an upper semi-continuous function of the parameter.

Corollary 4 (A simplest case of generalized eigenvalue problem). *Let K and M be compact operators, and let $L=I+M$. The point spectrum of $Kf=sLf$ (i.e. the set of those s for which there exist non-trivial f satisfying $Kf=sLf$) is one of the following:* 1) *the whole* C 2) $C-\{0\}$ 3) *a discrete set in C with at most one accumulation point at the origin.*

Proof. $Kf=sLf$ is deformed to $I+M-\frac{1}{s}K$, and the theorem may be applied. It is easily seen that all the three cases actually occur.

Reference

[1] I. F. Donin: Condition of triviality of deformations of holomorphic bundles on compact complex spaces. Math. Sbornic, **77** (119), No. 4, 602–623 (1968).