78. *An Extremal Property of the Polar Decomposition in von Neumann Algebras*

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1. In this paper, we shall concern with a polar decomposition of an operator in a von Neumann algebra in a connection with an extreme point of the unit ball of the algebra. Substantially, we shall show that an operator of a von Neumann algebra is the product of an extreme point of the unit ball and a positive operator in the algebra (Theorem 1).

As a few applications, we shall have a characterization of a finite von Neumann algebra and that every element of the unit ball of a von Neumann algebra is the average of two extreme points.

2. Let $\mathcal{H}$ be a Hilbert space. By an operator we shall mean a bounded linear operator acting on $\mathcal{H}$. For a C*-algebra $\mathcal{A}$ of operators, by $(\mathcal{A})_1$ we shall mean the *unit ball* of $\mathcal{A}$. An extreme point of $(\mathcal{A})_1$ will be called simply an *extreme point of* $\mathcal{A}$. Following after Halmos [5; p. 63] if U and V are partial isometries, write $U \leqq V$ in case V agrees with U on the initial space of U.

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all operators on $\mathcal{H}$, then every element in $\mathcal{L}(\mathcal{H})$ is the product of a maximal partial isometry (with respect to the above partial order) and a positive operator [5; p. 69]. A maximal partial isometry is an isometry or a co-isometry [5; p. 64]. By Kadison [6], for a factor, a necessary and sufficient condition that a partial isometry be an extreme point of the unit ball is that the partial isometry be an isometry or a co-isometry. Therefore, every operator on $\mathcal{H}$ has a representation as the product of an extreme point of $\mathcal{L}(\mathcal{H})$ and a positive operator.

Furthermore, let $\mathcal{A}$ be a finite von Neumann algebra on $\mathcal{H}$. It is essentially known that any element in $\mathcal{A}$ is the product of a unitary elment and a positive element of $\mathcal{A}$, and in finite factors, this fact is used repeatedly (e.g. [1], [4]). In a finite von Neumann algebra, the set of all extreme points of the unit ball is that of all unitary operators (cf. [2], [7], [10]). Therefore, any element of $\mathcal{A}$ is the product of an extreme point and a positive element.

We shall show the above fact is also true for a general von Neumann algebra:

Theorem 1. *Let $\mathcal{A}$ be a von Neumann algebra. Then any ele-*

ment A in $\mathcal{A}$ is represented by

$$A=VH,$$

where V is an extreme point of the unit ball of $\mathcal{A}$ and $H=(A^*A)^{\frac{1}{2}}$.

Proof. Let E and F be the support projections of A and A^*, respectively. Then there exists by [3; p. 334] a partially isometric operator U in $\mathcal{A}$ such that

$$A=UH,\ U^*U=E \quad \text{and} \quad UU^*=F.$$

For $1-E$ and $1-F$, applying the general comparability theorem [3; p. 228], we have a central projection G of $\mathcal{A}$ such that

$$(1-E)G\prec(1-F)G$$

and

$$(1-E)(1-G)\succ(1-F)(1-G).$$

Hence there exist two partially isometric operators V and W in $\mathcal{A}$ such that

$$V^*V=(1-E)G,$$
$$VV^*\leqq(1-F)G,$$
$$W^*W=(1-F)(1-G),$$

and

$$WW^*\leqq(1-E)(1-G).$$

Now, put $E_1=VV^*$, $E_2=WW^*$, and $V_0=U+V+W^*$. Then we have

$$V_0^*V_0=U^*U+V^*V+WW^*=E+(1-E)G+E_2$$

and

$$V_0V_0^*=UU^*+VV^*+W^*W=F+E_1+(1-F)(1-G).$$

Hence $V_0^*V_0$ and $V_0V_0^*$ are the sums of mutually orthogonal projections of $\mathcal{A}$, that is, V_0 is a partial isometry in $\mathcal{A}$. Since

$$V_0^*V_0=E+(1-E)G+E_2=G+E(1-G)+E_2\geqq G$$

and

$$V_0V_0^*=F+E_1+(1-F)(1-G)=(1-G)+FG+E_1\geqq 1-G,$$

by [1; Theorem 1] V_0 is an extreme point of $\mathcal{A}$.

On the other hand, we have

$$V_0H=UH+VH+W^*H=UH=A,$$

which completes the proof.

Here we shall give an application of Theorem 1. For $\mathcal{L}(\mathcal{H})$, extreme points of the unit ball are maximal partial isometries and vice versa. In a general von Neumann algebra $\mathcal{A}$, introducing an order structure among partial isometries of $\mathcal{A}$ as a substructure of partial isometries of $\mathcal{L}(\mathcal{H})$, as a corollary of Theorem 1, we have the following extension of the above fact:

Corollary 2. *Let $\mathcal{A}$ be a von Neumann algebra, then a necessary and sufficient condition that a partial isometry in $\mathcal{A}$ be maximal in $\mathcal{A}$ is that the partial isometry be an extreme point of the unit ball of $\mathcal{A}$.*

3. For a commutative B^*-algebra with the unit element, Phelps [8] (cf. also [11]) proved, that the unit ball is the uniformaly closed convex hull of the extreme points. For a general B^*-algebra, the same property is established by Russo and Dye [9]. On the other hand, the unit ball of $\mathcal{L}(\mathcal{H})$ is the convex hull of the extreme points (cf. [5; p. 265]). This suggests the following

Theorem 3. *Let $\mathcal{A}$ be a von Neumann algebra. If A is an element in the unit ball of $\mathcal{A}$, then there exist two extreme points V and W of the ball such that*

$$A=\frac{V+W}{2},$$

that is, A is the average of two extreme points of the unit ball.

Proof. By Theorem 1, we have a polar decompostion $A=UH$, where U is an extreme point of the unit ball and $H=(A^*A)^{\frac{1}{2}}$. Since H is hermitean and $\|H\|\leqq 1$, by [3; p. 4], there exists a unitary operator V in $\mathcal{A}$ such that

$$H=\frac{V+V^*}{2}.$$

Hence we have

$$A=UH=\frac{UV+UV^*}{2}.$$

It is clear that UV and UV^* are extreme points of the unit ball.

A necessary and sufficient condition that a von Neumann algebra $\mathcal{A}$ be finite is that only unitary elements of $\mathcal{A}$ be extreme points of the unit ball. By this result and Theorem 3, we have

Theorem 4 (Russo-Dye [9]). *Let $\mathcal{A}$ be a von Neumann algebra. $\mathcal{A}$ is finite if and only if the unit ball of $\mathcal{A}$ is the convex hull of all unitary elements of $\mathcal{A}$.*

4. In this section, we shall consider the property of the set of all regular elements of a von Neumann algebra $\mathcal{A}$.

Feldman and Kadison [4] determined the element contained in the closure of the set of all regular elements of a von Neumann algebra, and consequently pointed out that all regular elements of a II_1-factor are uniformly dense.

On the other hand, it is well known that the set of all regular elements in $\mathcal{L}(\mathcal{H})$ is uniformly dense if and only if $\mathcal{H}$ is finite dimensional, cf. [5; p. 70].

The following theorem is an extension of both cases:

Theorem 5. *Let $\mathcal{A}$ be a von Neumann algebra. Then $\mathcal{A}$ is finite if and only if the set of all regular elements of $\mathcal{A}$ is uniformly dense in $\mathcal{A}$.*

Proof. If $\mathcal{A}$ is finite, then an extreme point of $(\mathcal{A})_1$ is unitary. By Theorem 1, if A is an element of $\mathcal{A}$, then there is a unitary U in $\mathcal{A}$ such as

$$A=U|A|, \qquad |A|=(A^*A)^{\frac{1}{2}}.$$

For any $\varepsilon>0$, by the spectral theorem, there is a regular element $B\varepsilon\mathcal{A}$ such that $\||A|-B\|<\varepsilon$. Therefore, we have

$$\|A-UB\|=\|U|A|-UB\|\leqq\||A|-B\|<\varepsilon.$$

Since UB is regular, regular elements are uniformly dense in $\mathcal{A}$.

Suppose now that $\mathcal{A}$ is not finite, then there exists a partial isometry V in $\mathcal{A}$ such that $V^*V=1$ and $VV^*<1$, [3; p. 308]. Let A be an element of $\mathcal{A}$ such that $\|A-V\|<1$, then (as [5; p. 267])

$$\|1-V^*A\|=\|V^*(V-A)\|\leqq\|V-A\|<1.$$

Hence V^*A is regular in $\mathcal{A}$. If A were regular, then V^* is regular and a contradiction. Therefore, we have proved that the set of all regular elements of $\mathcal{A}$ can not be uniformly dense in $\mathcal{A}$ if $\mathcal{A}$ is not finite.

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