123. *Some Remarks on the Approximation of Nonlinear Semi-groups*

By Joseph T. CHAMBERS
Georgetown University, Washington, D. C.

(Comm. by Kinjirô KUNUGI, M. J. A., June 12, 1970)

1. Let X be a Banach space and U be a subset of X. Let $\{T(t)\,;\, t\geq 0\}$ be a family of nonlinear operators from U into itself satisfying the conditions:

(i) $T(0)=I$ (the identity mapping) and $T(t+s)=T(t)T(s)$ for $t, s\geq 0$.

(ii) For $x \in U$, $T(t)x$ is strongly continuous in $t\geq 0$.

(iii) $\|T(t)x-T(t)y\|\leq \|x-y\|$ for $x, y \in U$ and $t\geq 0$.

Such a family $\{T(t)\,;\, t\geq 0\}$ is called a nonlinear contraction semi-group on U. We define the infinitesimal generator A of the semi-group $\{T(t)\,;\, t\geq 0\}$ by

$$Ax-\lim_{h\to 0+}h^{-1}(T(h)-I)x$$

and the weak infinitesimal generator A' by

$$A'x=w\text{-}\lim_{h\to 0+}h^{-1}(T(h)-I)x$$

if the right sides exist. (The notation "lim" ("w-lim") means the strong limit (the weak limit) in X. We denote the domain of A by $D(A)$.)

H. F. Trotter [6] established the following result for linear contraction semi-groups.

Theorem. *Suppose that $\{T(t)\,;\, t\geq 0\}$ and $\{T'(t)\,;\, t\geq 0\}$ are linear contraction semi-groups of class (C_0) in the Banach space X with infinitesimal generators A and B, respectively. If $A+B$ (or its closure) is the infinitesimal generator of a semi-group $\{S(t)\,;\, t\geq 0\}$ of class (C_0), then*

$$S(t)x=\lim_{h\to 0+}(T(h)T'(h))^{[t/h]}x,\ x\in X.$$

[ ] *denotes the Gaussian bracket.*

In Section 2, we shall prove an extension of this theorem for the case of nonlinear contraction semi-groups on a subset U of a Banach space X. In Section 3, we shall approximate the semi-group $\{S(t)\,;\, t\geq 0\}$ by using $2^{-1}(T(2h)+T'(2h))$ which is the arithmetic mean of $T(2h)$ and $T'(2h)$. Note that, roughly speaking, $T(h)T'(h)$ may be regarded as the geometric mean of $T(2h)$ and $T'(2h)$.

2. The proofs in this paper are based upon the following theorem which was proved by I. Miyadera and S. Oharu [3], [4].

Theorem 2.1. *Let X be a Banach space and $X^{(k)}$ $(k=1,2,3,\cdots)$ be closed convex subsets of X.*

Suppose that $C_k\in \mathrm{Cont}(X^{(k)})$ (the contractions from $X^{(k)}$ into itself) $k=1,2,3,\cdots$, and that $\{h_k\}$ is a sequence such that $h_k>0$, $h_k\to 0$ (as $k\to\infty$).

If (i) $\lim h_k^{-1}(C_k-I)x=Ax,\ x\in D(\subset \bigcap_{k=1}^{\infty} X^{(k)})$,

(ii) *A (on D) is the restriction of the weak infinitesimal generator of a contraction semi-group $\{T(t);t\geq 0\}$ on a closed set $X^{(0)}$ (on a set $X^{(0)}$),*

(iii) *there exists a set $D_0(\subset D)$ such that for $x\in D_0$, $T(t)x\in D$ for a.a. $t\geq 0$,*

then for $x\in \overline{D_0}(x\in D_0)$

(2.1) *$T(t)x=\lim_{k\to\infty}C_k^{[t/h_k]}x$ uniformly in t on every bounded interval.*

Let X_1 and X_2 be subsets of a Banach space X and let $\{T(t);t\geq 0\}$ be a contraction semi-group on X_1 with infinitesimal generator A, and $\{T'(t);t\geq 0\}$ be a contraction semi-group on X_2 with infinitesimal generator B.

Theorem 2.2. *Let X_0 be a closed convex set such that $X_0\subset X_1\cap X_2$. Suppose that*

(i) *$T(t)X_0\subset X_0$ and $T'(t)X_0\subset X_0$ for $t\geq 0$,*

(ii) *$\lim_{h\to 0+}h^{-1}(T(h)T'(h)-I)x=Kx$ for $x\in D(\subset X_0\cap D(A)\cap D(B))$,*

(iii) *K is a restriction of the weak infinitesimal generator of a contraction semi-group $\{S(t);t\geq 0\}$ on a closed set X_3 (on a set X_3),*

(iv) *there exists a set $D_0\subset D$ such that if $x\in D_0$, then $S(t)x\in D$ for a.a. $t\geq 0$.*

Then for $x\in \overline{D_0}(x\in D_0)$

(2.2) *$S(t)x=\lim_{h\to 0+}\{T(h)T'(h)\}^{[t/h]}x$, uniformly in t on every bounded interval.*

Proof. Putting $C_h=T(h)T'(h)$ on X_0, we have that $C_h\in \mathrm{Cont}(X_0)$ for $h>0$ and $\lim_{h\to 0+}h^{-1}(C_h-I)x=Kx$ for $x\in D$. Hence, Theorem 2.1 (with $X^{(k)}=X_0, k=1,2,3,\cdots$, and $X^{(0)}=X_3$) implies that

$$S(t)x=\lim_{h\to 0+}C_h^{[t/h]}x\ \lim_{h\to 0+}\{T(h)T'(h)\}^{[t/h]}x$$

for $x\in \overline{D_0}(x\in D_0)$. Q.E.D.

Definition 2.1. A set-valued operator A in a Banach space X is said to be dissipative if for each $x,y\in D(A)$[1] and $x'\in Ax$ and $y'\in Ay$, there exists an $f\in F(x-y)$, F denotes the duality mapping between X and X^*, such that $\mathrm{re}\langle x'-y', f\rangle\leq 0$.

A is said to be maximal dissipative if A is not the proper restriction of any dissipative extension of A.

---

1) By $X\in D(A)$, we mean that $Ax\neq\emptyset$.

Theorem 2.3. *Let both X and X^* be uniformly convex and let X_0 be a closed convex set such that $X_0 \subset X_1 \cap X_2$.*

Suppose that

(i) $T(t)X_0 \subset X_0$ *and* $T'(t)X_0 \subset X_0$ *for* $t \geq 0$,

(ii) *A is maximal dissipative (as a set function),*

(iii) $A+B|_{X_0 \cap D(A) \cap D(B)}$ *is the infinitesimal generator of a contraction semi-group* $\{S(t); t \geq 0\}$ *on a closed set* X_3 *(on a set* X_3*).*

Then for $x \in \overline{X_0 \cap D(A) \cap D(B)} (x \in X_0 \cap D(A) \cap D(B))$

(2.3) $S(t)x = \lim_{h \to 0+} \{T(h)T'(h)\}^{[t/h]} x$ *uniformly in t on every bounded interval.*

Proof. First we prove that

(2.4) $\lim_{t \to 0+} t^{-1}(T(t)T'(t)x - x) = Ax + Bx$ for $x \in X_0 \cap D(A) \cap D(B)$.

Let $x \in X_0 \cap D(A) \cap D(B)$.

$$t^{-1}(T(t)T'(t)x - x) = t^{-1}(T(t)x - x) + t^{-1}(T(t)T'(t)x - T(t)x)$$

Since $\|t^{-1}(T'(t)x - x)\| \leq t^{-1}\int_0^t \|BT'(s)x\| ds \leq \|Bx\|$, we obtain

(2.5) $\|t^{-1}(T(t)T'(t)x - T(t)x\| \leq \|Bx\|$.

And, since $T(t) - I$ is dissipative,

$$\mathrm{re}\langle T(t)T'(t)x - T'(t)x - T(t)y + y, F(T'(t)x - y)\rangle \leq 0$$

hence

$$\mathrm{re}\langle z_t + t^{-1}(T(t)x - x) - t^{-1}(T'(t)x - x) - t^{-1}(T(t)y - y), F(T'(t)x - y)\rangle \leq 0$$

where $z_t = t^{-1}(T(t)T'(t)x - T(t)x)$.

Let $\{t_n\}$ be an arbitrary sequence such that $t_n \to 0+$. Since $\|z_{t_n}\| \leq \|Bx\|$ by (2.5), there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a $z \in X$ such that $z_{t_{n_k}} \rightharpoonup z$ (weak convergence). By the strong continuity of F, we have that

(2.6) $\mathrm{re}\langle z + Ax - Bx - Ay, F(x-y)\rangle \leq 0$ for all $y \in D(A)$.

The maximal dissipativity of A implies that $z - Bx = 0$, i.e., $z = Bx$. Therefore, $z_{t_{n_k}} \rightharpoonup Bx$, and hence $\|Bx\| \leq \liminf_{k \to \infty} \|z_{t_{n_k}}\|$. On the other hand, $\limsup_{k \to \infty} \|z_{t_{n_k}}\| \leq \|Bx\|$ by (2.5). So, we have $\|Bx\| = \lim_{k \to \infty} \|z_{t_{n_k}}\|$. The uniform convexity of X implies that $\lim_{t_{n_k} \to 0+} z_{t_{n_k}} = Bx$, so by the uniqueness of the limit we have that $\lim_{t \to 0+} z_t = Bx$. Consequently, we have (2.4). Now, setting $K = A + B|_{X_0 \cap D(A) \cap D(B)}$, we have that K is the infinitesimal generator of $\{S(t); t \geq 0\}$ and that $S(t)x \in D(K) \equiv X_0 \cap D(A) \cap D(B)$ for $t \geq 0$ and $x \in X_0 \cap D(A) \cap D(B)$ by Grandall and Pazy ([2]; Theorem 1.4). Therefore, the assumptions of Theorem 2.2 are satisfied with $K = A + B|_{X_0 \cap D(A) \cap D(B)}$ and $D_0 = D = X_0 \cap D(A) \cap D(B)$.

Q.E.D.

Remark 2.1. Let X and X^* be uniformly convex.

If $A + B|_{X_0 \cap D(A) \cap D(B)}$ is closed and $R(I - \eta(A + B|_{X_0 \cap D(A) \cap D(B)})) \supset X_0 \cap D(A) \cap D(B))$ for all $\eta > 0$, then $A + B|_{X_0 \cap D(A) \cap D(B)}$ is the infinitesimal generator of a contraction semi-group $\{S(t); t \geq 0\}$ on $X_0 \cap D(A) \cap D(B)$.[2] For details, see [5].

Corollary 2.1. *Let both X and X^* be uniformly convex, and let X_0 be a closed convex subset of X. Let $\{T(t); t \geq 0\}$ be a contraction semi-group on X with infinitesimal generator A and $\{T'(t); t \geq 0\}$ be a contraction semi-group on X with infinitesimal generator A* and $\{T'(t); t \geq 0\}$ be a contraction semi-group on X with infinitesimal generator B. *If A is maximal dissipative, $A+B$ is closed and $R(I-\eta(A+B)) \supset D(A+B)=D(A) \cap D(B)$ for all $\eta > 0$, then*

(i) *$A+B$ is the infinitesimal generator of a contraction semi-group $\{S(t); t \geq 0\}$ on $D(A) \cap D(B)$,*

(ii) *for each $x \in D(A) \cap D(B)$*

$$S(t)x = \lim_{h \to 0+} \{T(h)T'(h)\}^{[t/h]}x \quad \textit{uniformly}$$

in t on every bounded interval.

Remark 2.2. Theorem 2.3 is an extension of a result of Brezis and Pazy ([1]; Theorem 3.8) in their case X is a Hilbert space.

3. Let X_1 and X_2 be subsets of a Banach space X, and let $\{T(t); t \geq 0\}$ be a contraction semi-group on X_1 with infinitesimal generator A and $\{T'(t); t \geq 0\}$ be a contraction semi-group on X_2 with infinitesimal generator B.

Let X_0 be a closed convex set such that $X_0 \subset X_1 \cap X_2$. Then, we define for any $a, b \geq 0$ with $a+b > 0$ and $h > 0$, $C_h(a, b)$ on X_0 by

$$C_h(a,b) = \frac{aT((a+b)h) + bT'((a+b)h)}{a+b} \tag{3.1}$$

and set

$$A_h(a,b) = h^{-1}(C_h(a,b) - I). \tag{3.2}$$

Theorem 3.1. *Let $a, b \geq 0$ with $a+b > 0$ be arbitrary, but fixed. Suppose that*

(i) *$T(t)X_0 \subset X_0$ and $T'(t)X_0 \subset X_0$ for all $t \geq 0$,*

(ii) *$aA + bB|_{X_0 \cap D(A) \cap D(B)}$ is a restriction of the weak infinitesimal generator of a contraction semi-group $\{S_{a,b}(t); t \geq 0\}$ on a closed set X_3 (on a set X_3),*

(iii) *there exists a set $D_0 \subset D(\equiv X_0 \cap D(A) \cap D(B))$ such that if $x \in D_0$ then $S_{a,b}(t)x \in D$ for almost all $t \geq 0$.*

Then for $x \in \overline{D_0}(x \in D_0)$,

$$S_{a,b}(t)x = \lim_{h \to 0+} \left\{ \frac{aT((a+b)h) + bT'((a+b)h)}{a+b} \right\}^{[t/h]} x \tag{3.3}$$

uniformly in t on every bounded interval.

Proof. We first note that $C_h(a,b)$ is a contraction from X_0 into itself and that $A_h(a,b)x \to (aA+bB)x$ as $h \to 0+$ for $x \in D$ $(\equiv X_0 \cap D(A) \cap D(B))$. Hence, Theorem 2.1 (with $X^{(k)} = X_0$, $k=1,2,3,\cdots$, and $X^{(0)} = X_3$) implies that

2) "R" means "the range of".

$$S_{a,b}(t)x=\lim_{h\to 0+}C_h(a,b)^{[t/h]}x$$
$$\lim_{h\to 0+}\left\{\frac{aT((a+b)h)+bT'((a+b)h)}{a+b}\right\}^{[t/h]}x$$

for $x\in\overline{D_0}(x\in D_0)$. Q.E.D.

Corollary 3.1. *Let $a=b=1$. Then under the assumptions of the theorem, we have that*

$$S_{1,1}(t)x=\lim_{h\to 0+}\left\{\frac{T(2h)+T'(2h)}{2}\right\}^{[t/h]}x,\ x\in\overline{D_0} \tag{3.4}$$

($x\in D_0$) uniformly in t on every bounded interval.

Corollary 3.2. *Let $\{T(t);t\geq 0\}$ be a contraction semi-group on a closed convex subset X_0 with infinitesimal generator A. Suppose that there exists a set $D_0(\subset D(A))$ such that if $x\in D_0$ then $T(t)x\in D(A)$ for almost all $t\geq 0$. Then, for $x\in D_0$, $a,b\geq 0$ with $a+b>0$, we have*

$$T(at)x=\lim_{h\to 0+}\left\{\frac{aT((a+b)h)+bI}{a+b}\right\}^{[t/h]}x, \tag{3.5}$$

uniformly in t on every bounded interval.

Proof. In Theorem 3.1 put $X_1=X_2=X_0$ and let $\{T'(t);t\geq 0\}$ be the identity semi-group, i.e., $T'(t)\equiv I$ for $t\geq 0$. Then, the infinitesimal generator of $\{T'(t);t\geq 0\}$, B, is the zero operator and $D(B)=X_0$. Also, note that aA is the infinitesimal generator of the semi-group $\{T(at);t\geq 0\}$ on X_0. Q.E.D.

Finally, we present an application of Corollary 3.2. In (3.5) set $a=\xi$ and $b=1-\xi$ for $0\leq\xi\leq 1$; $t=1$ and $1/h=n$, then we have

$$T(\xi)x=\lim_{n\to\infty}((1-\xi)I+\xi T(1/n))^n x \text{ for } 0\leq\xi\leq 1 \tag{3.8}$$

and $x\in\overline{D_0}$.

Remark 3.1. The representation (3.8) holds uniformly on $[0,1]$ for $x\in\overline{D(A)}$ without the assumption of the set D_0 in the Corollary 3.2. This is proved in a more direct way by I. Miyadera and S. Oharu in [4].

Example (Linear Case). Let $C[0,\infty]$ be the set of all continuous functions $x(\cdot)$ defined on $[0,\infty]$ such that $\lim_{t\to\infty}x(t)$ exists. Then, $C[0,\infty]$ equipped with the supremum norm is a Banach space. Let $\{T(t);t\geq 0\}$ be the semi-group of right translations on $C[0,\infty]$, i.e., $(T(t)x)(s)=x(t+s)$ for $t\geq 0$ and $x(\cdot)\in C[0,\infty]$. Hence, $\{T(t);t\geq 0\}$ is a linear contraction semi-group on $C[0,\infty]$ and $\overline{D(A)}=C[0,\infty]$.

Using (3.8) with $D_0=D(A)$, we have that for $x(\cdot)\in[0,\infty]$

$$\begin{aligned}[T(\xi)x](s)&=\lim_{n\to\infty}([(1-\xi)I+\xi T(1/n)]^n x)(s)\\&=\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}(1-\xi)^{n-k}\xi^k(T(k/n)x)(s)\\&=\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}(1-\xi)^{n-k}\xi^k x(s+k/n)\end{aligned} \tag{3.9}$$

Putting $s=0$ in (3.9), we get

$$x(\xi)=\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}(1-\xi)^{n-k}\xi^k x(k/n),\ 0\leq\xi\leq 1. \tag{3.10}$$

Note that (3.10) gives Berstein's Approximation Theorem.

Finally, the author wishes to express his gratitude to Professor Isao Miyadera for his valuable advice.

References

[1] H. Brezis and A. Pazy: Semi-groups of nonlinear contractions on convex sets (to appear).

[2] M. G. Crandall and A. Pazy: Semi-groups of nonlinear contractions and dissipative sets. Jour. of Functional Analysis, **3**, 376–418 (1969).

[3] I. Miyadera and S. Oharu: Approximation of semi-groups of nonlinear operators. Tôhoku Math. Jour., **22**, 24–47 (1970).

[4] ——: Approximation of semi-groups of nonlinear operators. II (to appear).

[5] S. Oharu: On the generation of semi-groups of nonlinear contractions (to appear).

[6] H. F. Trotter: On the product of semi-groups of operators. Proc. Amer. Math. Soc., **10**, 545–551 (1959).