209. A Note on C-compact Spaces

By Shozo SAKAI
Department of Mathematics, Shizuoka University

(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 12, 1970)

According to G. Viglino [7], a topological space $(X, \mathcal{T})$ is said to be *C-compact* if given a closed set A of X and a $\mathcal{T}$-open covering $\mathcal{U}$ of A, there is a finite number of elements of $\mathcal{U}$, say $U_i, 1 \leqq i \leqq n$, with $A \subset \bigcup_{i=1}^{n} \bar{U}_i$. It was shown by Viglino that in Hausdorff spaces the following implications hold and neither of them is reversible:

compact $\Rightarrow$ C-compact $\Rightarrow$ minimal Hausdorff.

Here a space X is *minimal Hausdorff* if X is Hausdorff and each open filter-base on X (i.e. a filter-base composed exclusively of open sets of X) with a unique adherent point is convergent.

The main results of this note are that (1) the product of a C-compact space and a compact space need not be C-compact in general, and that (2) there exist minimal Hausdorff spaces of arbitrary infinite cardinality which are not C-compact.

Theorem 1. *For any topological space X, the following properties of X are equivalent*:

(1) *X is C-compact,*

(2) *if A is a closed set of X and $\mathcal{F}$ a family of closed sets of X with $\cap \mathcal{F} \cap A = \emptyset$, then there is a finite number of elements of $\mathcal{F}$, say F_i, $1 \leqq i \leqq n$, with $\bigcap_{i=1}^{n}(\mathrm{Int}\, F_i) \cap A = \emptyset$.*

(3) *if A is a closed set of X and $\mathcal{G}$ an open filter-base on X whose elements have non-empty traces with A, then there is an adherent point of $\mathcal{G}$ in A.*

Proof. (1) $\Rightarrow$ (2). Let A be a closed subset of a C-compact space X and $\mathcal{F}$ a family of closed sets of X with $\cap \mathcal{F} \cap A = \emptyset$. Since $\mathcal{U} = \{X - F \mid F \in \mathcal{F}\}$ is a family of open sets of X covering A, there is a finite number of elements of $\mathcal{U}$, say $U_i = X - F_i$, $1 \leqq i \leqq n$, with $\bigcup_{i=1}^{n} \bar{U}_i \supset A$. Therefore, $\bigcap_{i=1}^{n}(\mathrm{Int}\, F_i) = X - \bigcup_{i=1}^{n} \bar{U}_i \subset X - A$.

(2) $\Rightarrow$ (3). Assume that there exist a closed set A and an open filter-base $\mathcal{G}$ on X having no adherent point in A whose elements have non-empty traces with A. Since $\mathcal{F} = \{\bar{G} \mid G \in \mathcal{G}\}$ is a family of closed sets of X with $\cap \mathcal{F} \cap A = \emptyset$, there is a finite number of elements of $\mathcal{F}$, say $F_i = \bar{G}_i$, $1 \leqq i \leqq n$, with $\bigcap_{i=1}^{n}(\mathrm{Int}\, F_i) \cap A = \emptyset$. Then we have $\bigcap_{i=1}^{n} G_i \cap A = \emptyset$. Since $\mathcal{G}$ is a filter-base, there is an element $G \in \mathcal{G}$ with $G \cap A = \emptyset$. This contradicts the assumption on $\mathcal{G}$.

(3) $\Rightarrow$ (1). Assume that X is not C-compact. There are a closed

set A and a covering $\mathcal{U}$ of A consisting of open sets of X such that for any finite number of elements of $\mathcal{U}$ their closures do not cover A. Since $\mathcal{G}=\{X-\bigcup_{i=1}^{n}\bar{U}_{\lambda_i}\mid n$ is finite, $U_{\lambda_i}\in\mathcal{U}\}$ is an open filter-base on X whose elements have non-empty traces with A, there is an adherent point x of $\mathcal{G}$ in A. Then $x\in\bar{G}$ for each $G\in\mathcal{G}$. Particularly, $x\in\overline{X-\bar{U}}=X-U$ for each $U\in\mathcal{U}$. Therefore, $\mathcal{U}$ is not a covering of A, a contradiction. This completes the proof of Theorem 1.

Remark. In (3) of Theorem 1, if we replace $\mathcal{G}$ by "open filter-base on A", then each closed subspace of X is H-closed (absolutely closed) under the condition of X being Hausdorff. A Hausdorff space with this property is compact by Katětov [4].

Theorem 2. *If the product $\prod X_\lambda$ of non-empty topological spaces X_λ is C-compact, then so is X_λ for each λ.*

Proof. Since the continuous image of a C-compact space is C-compact, this is trivial.

In [7], Viglino asked whether the product of C-compact spaces is C-compact or not. The following Example 1 answers this question.

Example 1. *There exist a C-compact Hausdorff space X and a compact Hausdorff space Y such that $X\times Y$ is not C-compact.* Let X be an example due to Viglino which is C-compact Hausdorff but not compact. Since this example is necessary for later results, we will describe it. Let

$X=\{(a,b)\mid a=1/n,\ b=1/m$ or $a=1/n,\ b=0$ or $a=0,\ b=0;\ n,m\in N\}$,

where N stands for the set of all positive integers. To describe the topology of X, partition N into infinitely many infinite disjoint classes, $\{N_i\mid i\in N\}$. Define subsets of X as follows:

$$H_{ik}=\{(1/i,0)\}\cup\{(1/i,1/m)\mid m\geqq k\}\cup\{(1/n,1/m)\mid n\geqq k,\ m\in N_i\},$$

$$L_k=\{(0,0)\}\cup\{(1/n,1/m)\mid n>k,\ m\notin N_i,\ 1\leqq i\leqq k\}.$$

Let $\mathcal{T}$ be the topology of X generated by

$$\{\{(1/n,1/m)\}\mid n,m\in N\}\cup\{H_{ik}\mid i,k\in N\}\cup\{L_k\mid k\in N\}.$$

Then $(X,\mathcal{T})$ is a C-compact Hausdorff but not compact space. Let $Y=\{y_0,y_1,y_2,\cdots\}$ be a one-point compactification of a countable discrete space $\{y_1,y_2,\cdots\}$. Consider $A=\{(1/n,0;y_n)\mid n\in N\}$ in $X\times Y$. It is easily proved that A is closed in $X\times Y$. $\mathcal{U}=\{H_{nk(n)}\times\{y_n\}\mid n\in N\}$ is a covering of A consisting of open sets of $X\times Y$. Since there is no finite number of elements of $\mathcal{U}$ whose closures cover A, $X\times Y$ is not C-compact.

Let us say that a space X has Property (*) if every continuous function from X into a Hausdorff space is closed. Viglino proved that each C-compact space has Property (*) and asked whether a Hausdorff space having Property (*) is C-compact or not. Since the image of A in Example 1 by projection $X\times Y\to Y$ is not closed, $X\times Y$ in Example 1 has not Property (*). Therefore, Property (*) of a topological space is not productible.

Each C-compact subspace of a Hausdorff space is closed by Property (*), and there exists a closed, not H-closed subspace in a C-compact Hausdorff space (for instance, consider $\{(1/n, 0) \mid n \in N\}$ in X in Example 1). While a regularly closed subspace in a H-closed space is H-closed [2], and a closed and open subspace of a C-compact space is C-compact [7]. The following Example 2 shows that even a regularly closed subspace of a C-compact space need not be minimal Hausdorff.

Example 2. *There exist a C-compact Hausdorff space X and a regularly closed subspace A of X such that A is not minimal Hausdorff.* Let X be a C-compact Hausdorff not compact space in Example 1. Take i and k in N with $i<k$, and let $A=\bar{H}_{ik}=H_{ik}\cup\{(1/n, 0) \mid n \geqq k\}$. Then A is regularly closed in X. Let U be a regularly open subset of A containing $(1/i, 0)$. Then there exists k_0 in N such that $(1/n, 0) \in U$ for $n \geqq k_0$. Thus for an open neighborhood H_{ik} of $(1/i, 0)$ in A, there is no regularly open set U with $(1/i, 0) \in U \subset H_{ik}$. Therefore, A is not semi-regular [5]. Since a space is minimal Hausdorff if and only if it is H-closed and semi-regular [4], A is not minimal Hausdorff.

Viglino's example of minimal Hausdorff but not C-compact space is uncountable. We will show the existence of minimal Hausdorff but not C-compact spaces for arbitrary infinite cardinality.

Example 3. *There exists a countable minimal Hausdorff space which is not C-compact.* Let $X=\{a_{ij}, b_{ij}, c_i, a, b \mid i, j \in N\}$ be an example due to Urysohn [6] which is minimal Hausdorff but not compact, see [1] for details. In X, $A=\{c_i \mid i \in N\}$ is a closed set and

$$\mathcal{U}=\{U_i=\{a_{ij}, b_{ij}, c_i \mid j=n_i, n_i+1, \cdots\} \mid i \in N\}$$

is a covering of A consisting of open sets of X where n_i are integers. Since there is no finite number of elements of $\mathcal{U}$ whose closures cover A, X is not C-compact.

Example 4. *There exist minimal Hausdorff spaces of arbitrary infinite cardinality which are not C-compact.* Let X be a countable minimal Hausdorff but not C-compact space. Given an infinite cardinal K, take Y to be any compact Hausdorff space of cardinal K (for instance, the one-point compactification of a discrete space of K points). Since X and Y are minimal Hausdorff, $X \times Y$ is minimal Hausdorff by Ikenaga [3] and is obviously of cardinal K. By Theorem 2, $X \times Y$ is not C-compact.

After this manuscript had been written, the author found the review of "C-compact spaces" written by Viglino himself in Zentralblatt für Mathematik und ihre Grenzgebiete, **185**, 507 (1970). In his review, Viglino reported that the product of a C-compact space with a closed unit interval need not be C-compact.

References

[1] M. P. Berri: Minimal topological spaces. Trans. Amer. Math. Soc., **108**, 97–105 (1963).
[2] N. Bourbaki: Topologie générale, 4th ed. Actualités Sci. Ind., no. 1142, Hermann, Paris (1965).
[3] S. Ikenaga: Products of minimal topological spaces. Proc. Japan Acad., **40**, 329–331 (1964).
[4] M. Katětov: Über H-abgeschlossene und bikompakte Räume. Časopis Pěst. Mat. Fys., **69**, 36–49 (1940).
[5] M. H. Stone: Applications of the theory of Boolean rings to general topology. Trans. Amer. Math. Soc., **41**, 375–481 (1937).
[6] P. Urysohn: Über die Mächtigkeit der zusammenhängenden Mengen. Math. Ann., **94**, 266–295 (1925).
[7] G. Viglino: C-compact spaces. Duke Math. J., **36**, 761–764 (1969).